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SPECTRAL DECOMPOSITION OF MATRICES FOR HIGH SCHOOL STUDENTS

ALBERT WILANSKY, Lehigh University

This topic has developed over a period of seven years in a summer "Junior Research" program sponsored by the National Science Foundation. The participants were high school students, mostly about to be seniors, who were selected on the basis of their mathematical talent. The author was assisted at various times by I. D. Berg, A. K. Snyder, G. A. Stengle, and T. M. Morrisette. (There are obvious reasons for not mentioning the names of the students, even those who made quite good contributions.) The development of the subject may be judged by the early report [**10**].

Even in the context of mathematics available at this level, several problems have arisen naturally that seem intractable by any standards, and many meaningful research projects were carried out by the students. Some of these will be described below.

Mathematicians will recognize special cases of the spectral theorem (here not restricted to normal matrices) and some properties of field extensions. The final results obtained may be extracted by suitable specialization from those of [**3**], [**6**], [**7**], [**8**], and [**9**]; however, the techniques of these articles are quite different from the ones developed here all of which have been successfully presented to (and partially developed by) high school students.

1. Residue rings. We denote by J_n the familiar residue ring mod n. For example, $J_6=\{0, 1, 2, 3, 4, 5\}$. We write $2+3=5$, $4+5=3$ (rather than $4+5\equiv 3 \pmod 6$), $4\times 5=2$ and so on. In J_6 we write $8=2$, $14=2$; for example $3+4=7=1$.

Each J_n is a commutative ring with identity. If $a\cdot b=1$ we say that a and b are *reciprocals*; the Euclidean algorithm shows that a member x of J_n has a reciprocal in J_n if and only if x and n have no common factor larger than 1 in the ordinary arithmetic of the positive integers. In particular, if p is prime, J_p is a field (a commutative ring with identity in which every nonzero member has a reciprocal). We shall make use of Fermat's Theorem to the effect that to each nonzero $a\in J_p$ corresponds a smallest $k>0$ such that $a^k=1$, and $k\mid(p-1)$. (See [**2**], pp 125, 131.) Any solution of the equation $x^2=x$ is called an *idempotent*.

2. Matrices. We shall be concerned mostly with 2 by 2 matrices with entries in some residue ring J_n. These will be referred to as matrices over J_n, and the collection of all of them will be written $M(J_n)$. As usual

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$
$$= \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}, k\cdot\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

The search for idempotent matrices motivates an attempt to simplify the square of a matrix. Setting

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}; \ \Delta = ad - bc \text{ (the } \textit{determinant}\text{)}; \ t = a + d \text{ (the } \textit{trace}\text{)};$$

we have

$$A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix} = \begin{bmatrix} a^2 + ad - \Delta & bt \\ ct & d^2 + ad - \Delta \end{bmatrix}$$

$$= \begin{bmatrix} at & bt \\ ct & dt \end{bmatrix} - \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix} = tA - \Delta I.$$

The equation $A^2 - tA + \Delta I = 0$ is called the *characteristic equation* of A. An obvious sufficient condition that A be idempotent is $t = 1$ and $\Delta = 0$. This condition is not necessary.

$$\left(\text{Witness } \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \text{ over } J_6, \text{ not to mention the zero and identity matrices.} \right)$$

THEOREM 1. *Let $A \in M(J_p)$, p prime, $A \neq 0$, $A \neq I$. Then A is idempotent if and only if $t = 1$, $\Delta = 0$.*

We have $A^2 - tA + \Delta I = 0$, so that the condition is obviously sufficient. Conversely, suppose that A is idempotent and that $t \neq 1$. The characteristic equation says $A - tA + \Delta I = 0$; since J is a field this yields $A = [-\Delta/(1-t)]I$ which we abbreviate as $A = \lambda I$. Since A is idempotent, so is λI; that is, $\lambda^2 = \lambda$. Either $\lambda = 0$, or if not, multiplying by $1/\lambda$ yields $\lambda = 1$. Thus either $A = 0$ or I, both excluded. It follows that $t = 1$ after all. Setting $A^2 = A$ and $t = 1$ in the characteristic equation yields $\Delta I = 0$; hence $\Delta = 0$. (It would also be easy to check that $\Delta(A^2) = (\Delta A)^2$; hence Δ is idempotent; hence $\Delta = 0$ or else A has an inverse forcing $A = I$.)

3. Eigenvalues. Let $A \in M(J_n)$ have trace t, determinant Δ. Any solution of the equation $x^2 - tx + \Delta = 0$ is called an *eigenvalue* of A. For example, define $A \in M(J_8)$ by

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}.$$

Then $t = 0$, $\Delta = 7$ and the characteristic equation is $A^2 + 7I = 0$. The solutions of the equation $x^2 + 7 = 0$ in J_8 are 1, 3, 5, 7 so A has 4 eigenvalues. Considering that, in J_8, $x^2 + 7 = (x-1)(x-7) = (x-3)(x-5)$, it is reasonable to list the eigenvalues as 1, 7; 3, 5. On the other hand, let

$$B = \begin{bmatrix} 1 & 1 \\ 5 & 7 \end{bmatrix} \in M(J_8).$$

Then B has no eigenvalues, for since $t = 0$, $\Delta = 2$, we are looking for solutions of $x^2 \pm 2 = 0$; but the squares of 0, 1, 2, 3, 4, 5, 6, 7 are 0, 1, 4, 1, 0, 1, 4, 1. Thus $x^2 \neq -2$ for all $x \in J_8$. (Note: $-2 = 6$.)

THEOREM 2. *Let p be a prime and $A \in M(J_p)$. Then A has two, one, or no eigenvalues.*

The upper bound, two, follows from the fact that a quadratic equation over a field can have at most two roots. (See, for example [4], Lemma 5.2.)

THEOREM 3. *Let $A \in M(J_p)$ with p prime, and let α, β be the eigenvalues of A. Then $(A-\alpha I)(A-\beta I)=0$. This holds even if $\alpha=\beta$.*

Since α, β are solutions of $x^2-tx+\Delta=0$ we have $x^2-tx+\Delta=(x-\alpha)(x-\beta)$. Thus $\alpha+\beta=t$, $\alpha\beta=\Delta$ and so $(A-\alpha I)(A-\beta I)=A^2-(\alpha+\beta)A+\alpha\beta I=A^2-tA+\Delta I=0$.

4. Statement of the problem. (The following result may be greatly generalized; see [1].)

THEOREM 4. *Let x be a member of a finite ring. Then there exists a positive integer k such that x^k is idempotent.*

The sequence $\{x, x^2, x^3, \cdots\}$ must have repetition, say $x^r=x^s$, $r>s$. Let $d=r-s$. Choose an integer t with $td>s$ and let $u=td-s$. Then x^{s+u} is idempotent, as we see by the following computation: $x^s=x^{s+d}=x^{s+2d}=\cdots=x^{s+td}$, hence $x^{s+u}=x^{s+td+u}$; but $s+td+u=2(s+u)$.

DEFINITION. *The order of x is the least positive integer k such that x^k is idempotent.*

This generalizes the usual definition which is given for nonzero elements of a field, namely the least positive integer k such that $x^k=1$. We are assured by Theorem 4 that every element of a finite ring has an order. For example, in J_{15}, the order of 3 is 4 since $3^2=9$, $3^3=12$ are not idempotent but $3^4=6$ is since $6\times 6=6$.

COROLLARY. *Every member of $M(J_n)$ has an order.*

For there are only finitely many 2 by 2 matrices over J_n; n^4 of them to be exact.

PROBLEMS. 1. What is the order of a given matrix?
2. What orders are possible for any 2 by 2 matrix over J_n?

In the early stages of this problem, direct computation produced the maximum order 3 for $n=2$, and 8 for $n=3$. Since $M(J_5)$ has 625 matrices, this approach had to be abandoned. There were much excitement and conjecture in the class over the possibilities for the maximum order for $n=5$.

A few computations will illustrate some primitive methods of computing an order. These yield results (eventually) but very little insight. Let

$$A=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\in M(J_3);$$

then A is not idempotent ($t=2\neq 1$) but

$$A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

is idempotent ($t=4=1$, $\Delta=3=0$); hence the order of A is 2. Let

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in M(J_3);$$

then $B, B^2, \cdots B^7$ are not idempotent, but $B^8=I$, so the order of B is 8. (Notice that if Δ has a reciprocal in J_n and k is the order of A, then $A^k=I$ since ΔA^k is a unit and A^k is idempotent.) Some students listed all 81 two by two matrices over J_3 with their orders. These are 1, 2, 3, 4, 6, 8. Further investigation revealed the curious fact that the matrices of order 4, 8 have no eigenvalues, those of order 3, 6 have one, and those of order 1, 2 have two eigenvalues, with the exception of 0, I. The staff was unable to "explain" this at first, but subsequent developments, described below, clarified the situation completely, yielding an exact prediction of orders in $M(J_n)$ for all n.

The technique just given may be improved. Let

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in M(J_3).$$

Then $B^2=B+I$ (the characteristic equation),

$$\begin{aligned} B^3 &= B\cdot B^2 = B(B+I) = B^2+B = (B+I)+B \\ &= 2B+I,\ B^4 = B\cdot B^3 = 2B^2+B = 2(B+I)+B \\ &= 2I,\ (\text{since } 3=0),\ B^5 = 2B,\ B^6 = B^4\cdot B^2 \\ &= 2I(B+I) = 2B+2I,\ B^7 = B^4\cdot B^3 = 2B^3 \\ &= B+2I,\ B^8 = (B^4)^2 = I \end{aligned}$$

Now the traces of B^2, B^5, B^6, B^7 are 0, 2, 0, 2 so they are not idempotent; further $B\neq B^2$, $B^3\neq B^6$, $B^4\neq B^8$ so B, B^3, B^4 are not idempotent. Thus the order of B is 8.

5. Resolution of the identity. Consider the three members 6, 10, 15 of J_{30}. They are idempotents, $6+10+15=1$, and $6\times 10=6\times 15=10\times 15=0$.

DEFINITION. *A set* $(a_1, a_2, \cdots, a_n)$ *is called a resolution of the identity if each* a_i *is idempotent and not* 0 *or* 1, $a_1+a_2+\cdots+a_n=1$, *and* $a_ia_j=0$ *if* $i\neq j$.

An example of a matrix resolution of the identity is

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix},\ B = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix},\ C = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \text{ over } J_6.$$

(Here $A+B+C=I$.)

THEOREM 5. *In* $M(J_p)$, *p prime, a resolution of the identity has exactly two members.*

This is immediate from Theorem 1 and the fact that the trace of I is 2.

6. Spectral decomposition. There is a certain class of matrices for which computation is very easy, with the result that the orders are easy to obtain. Suppose that (E, F) is a matrix resolution of the identity; that is $E^2=E$, $F^2=F$, $EF=FE=0$, $E+F=I$, and let α, β be members of the ring of entries of E, F. (For example, if E, F are matrices over J_5, then $\alpha, \beta \in J_5$.) Now set

$$A = \alpha E + \beta F. \tag{1}$$

Then $A^2=\alpha^2E^2+\alpha\beta EF+\beta\alpha FE+\beta^2F^2=\alpha^2E+\beta^2F$. Similarly $A^3=\alpha^3E+\beta^3F$ and, by easy induction,

$$A^n = \alpha^n E + \beta^n F, \qquad n = 1, 2, 3, \cdots. \tag{2}$$

Although irrelevant for order questions, it is interesting to ask if equation (2) holds for other n. For example, let us try it for $n=1/2$. If α, β have square roots, say $\alpha^{1/2}$, $\beta^{1/2}$, set $B=\alpha^{1/2}E+\beta^{1/2}F$ and we soon see that $B^2=(\alpha^{1/2})^2E+(\beta^{1/2})^2F=A$, and so indeed B is a square root of A. For $n=0$, equation (2) reads (with reasonable interpretation) $I=E+F$, and this also is true. Similarly equation (2) holds for negative n if we take A^{-1} as the inverse of A.

If a matrix A has a representation in the form (1), with $\alpha\neq\beta$, and with E, F a resolution of the identity, we say that A has a *spectral decomposition.* (If we allowed $\alpha=\beta$ we would have $A=\alpha(E+F)=\alpha I$, a trivial case.)

THEOREM 6. *Let $A\in M(J_p)$, p prime, and suppose that A has a spectral decomposition (1). Then α, β are the eigenvalues of A, and α, β, E, F are uniquely determined by A.*

From (1) it follows that $EA=AE=\alpha E$ since $EF=0$. Also $EA^2=(EA)A=\alpha EA=\alpha^2E$. Thus if we multiply the characteristic equation $A^2-tA+\Delta I=0$ by E we obtain $\alpha^2E-t\alpha E+\Delta E=0$. Since $E\neq 0$ this yields $\alpha^2-t\alpha+\Delta=0$, and so α is an eigenvalue. (This argument works even if $\alpha=\beta$.) Similarly β is an eigenvalue. This shows that α, β are uniquely determined (using Theorem 2). To see that E and, similarly, F, are uniquely determined by (1), replace F by $I-E$ obtaining $A=(\alpha-\beta)E+\beta I$. Thus

$$E = \frac{A - \beta I}{\alpha - \beta} \tag{3}$$

$$F = \frac{\alpha I - A}{\alpha - \beta} \tag{4}$$

7. Solution of the problem for spectrally decomposed matrices over J_p. Let p be a prime and suppose that $A\in M(J_p)$ has a spectral decomposition as in equation (1) with $\alpha\neq\beta$. If $\alpha=0$, then $\beta\neq 0$; let k be the order of β. Fermat's Theorem shows that $k\mid(p-1)$. We have $A^r=\beta^rF$ for all r; thus $A^k=F$ and the order of A is k. Any divisor of $p-1$ can be obtained in this way as an order for $A\in M(J_p)$. Next suppose $\alpha\neq 0$ and $\beta\neq 0$. Let u and v be the orders of α and β respectively. Then the order of A is m, the least common multiple of α and β for we have $A^m=\alpha^mE+\beta^mF=E+F=I$ is idempotent, while if A^r is idempotent, we have $A^r = A^{2r}$; thus by Theorem 6, $\alpha^r = \alpha^{2r}$, $\beta^r = \beta^{2r}$ and so r is a multiple of both u and v. Once again we find that the order of A is a factor of $p-1$.

8. Solution of the problem for matrices with two eigenvalues over J_p, p prime. This is accomplished by the pleasant device of showing that every such matrix has a spectral decomposition. Let α, β be its eigenvalues and define E, F by formulas (3), (4). Then $EF=0$ by Theorem 3; $E+F=I$ is a trivial computation, $E^2=E(I-F)=E-EF=E$, and, similarly, $F^2=F$. Thus (E, F) is a resolution of the identity. Finally, it is easy to compute that $\alpha E+\beta F=A$. This shows that A has a spectral decomposition and reduces the problem to that solved in Section 7.

9. Solution of the problem for matrices with repeated eigenvalues over J_p, p prime. We invent a type of spectral decomposition. Let α be the only eigenvalue of $A \in M(J_p)$. By Theorem 3, $(A-\alpha I)^2=0$ so if we set $N=A-\alpha I$ we shall have

$$A = \alpha I + N, \quad \text{with} \quad N^2 = 0. \tag{5}$$

The point of this decomposition is that we obtain $A^2=\alpha^2 I+2\alpha N$, since $N^2=0$; $A^3=\alpha^3 I+3\alpha^2 N$, and, in general,

$$A^r = \alpha^r I + r\alpha^{r-1} N. \tag{6}$$

THEOREM 7. *Let $A \in M(J_p)$, p prime and suppose that $A=\alpha I+N$ with $N^2=0$, $\alpha \neq 0$. Then A is idempotent if and only if $\alpha=1$ and $N=0$.*

If $A=A^2$, then $\alpha I+N=\alpha^2 I+2\alpha N$; hence $(\alpha-\alpha^2)I=(2\alpha-1)N$. Squaring both sides yields $(\alpha-\alpha^2)^2=0$; hence $\alpha=\alpha^2$ and so $\alpha=1$. The same equation then yields $N=0$.

Now if A has the form (5) with $\alpha \neq 0$, suppose that the order of α is k, and that of A is m. Applying Theorem 7 to (6) yields $\alpha^m=1$, $m\alpha^{m-1}=0$. The first equation says m is a multiple of k; the second becomes $m=0$ after multiplying by α and thus m is a multiple of p. Conversely, if m is a multiple of k and p, (6) yields $A^m=I$; thus m is the least common multiple of p and k. Since k may be any divisor of $p-1$ we have all kp with $k \mid (p-1)$ as the orders of matrices with only one eigenvalue (repeated eigenvalues, an eigenvalue of multiplicity 2).

For $M(J_3)$, $k=1$ or 2, and so 3, 6 are the orders of matrices with one eigenvalue.

10. Quadratic extensions. The case of a matrix with no eigenvalues requires some completely new techniques. We were lucky enough to recognize this in an unrelated area which had been studied for its own merits for a few summers. Let p be a prime and let g be a quadratic polynomial with leading coefficient 1 and with no root in J_p. For example, $g(v)=v^2-2$ has no root in J_3. Let $J_{p,g}$ be the set of all first degree polynomials in an indeterminate v, coefficients in J_p; thus the members of $J_{p,g}$ are expressions $a+bv$ with a, $b \in J_p$. Addition is defined by $(a+bv)+(c+dv)=a+c+(b+d)v$, and multiplication by ordinary multiplication over J_p but with $g(v)$ taken to be 0. For example, let $g(v)=v^2-2$ and operate in $J_{3,g}$. Then $(2+v)(1+2v)=2+5v+2v^2=2+2v+2v^2$ (since $5=2$ in J_3) $=2+2v+2(v^2-2)+4=2v$ since $v^2-2=0$, and $2+4=0$. (Algebraists will recognize this as a simple extension of the field J_p by a root of g. See [2], p. 367.) In this computation we came upon $2+2v+2v^2$; a more systematic procedure for seeing that this is $2v$ is to divide it by v^2-2 using ordinary long division (in J_p) and keep only the remainder, $2v$. This is analogous to obtaining $7=2$ in J_5, either by writing $7=5+2=0+2=2$ or by dividing 7 by 5 and keeping only the remainder 2. (This procedure can be applied to any ring. Taking $g(v)=v^2+1$ and defining R_g as the set of all $a+bv$, a, b real, we see that R_g is the complex numbers. For example $v^2=-1$ since $v^2=v^2+1-1=0-1=-1$.) Notice that $J_p \subset J_{p,g}$ since $a=a+0\cdot v$.

LEMMA 8. *$J_{p,g}$ is a field.*

The only nontrivial part of this result is that $a+bv$ has a reciprocal if $a+bv\neq 0$ (i.e., $a\neq 0$ or $b\neq 0$). If $b=0$, $a+bv$ has $1/a$ as its reciprocal. Now suppose $b\neq 0$. Suppose that $g(v)=v^2+sv+t$; let $z=bv+bs-a$, and let $w=(a+bv)z$. Then $w=abv+abs-a^2+b^2v^2+b^2sv-abv=b^2(v^2+sv)+abs-a^2=-b^2t+abs-a^2$ $=-b^2(\lambda^2+s\lambda+t)$ where $\lambda=-a/b$. Thus $w=-b^2g(\lambda)\neq 0$ by our hypothesis about g. Moreover $w\in J_p$. It follows that $(a+bv)\cdot(z/w)=1$. (For more insight on this proof, see [2], corollary, p. 369.)

LEMMA 9. *Let p be a prime, $p\neq 2$, and n a nonsquare in J_p. Let $g(v)=v^2-n$. Then every member of J_p has a square root in $J_{p,g}$.*

Every square in J_p has a square root in J_p. Next let m be a nonsquare in J_p. Then $(m/n)^{(p-1)/2}=(-1)/(-1)=1$ and so m/n is a square in J_p. (See [5], p. 38, Theorem 17.4.) Say $m/n=r^2$. Then $(rv)^2=r^2v^2=r^2n=m$.

LEMMA 10. *With p, g as in Lemma 9 every quadratic equation over J_p has a solution in $J_{p,g}$.*

Let the quadratic equation be $ax^2+bx+c=0$, $a\neq 0$. This has the solution $x=(-b+(b^2-4ac)^{1/2})/2a$ and Lemma 9 assures us that b^2-4ac has a square root in $J_{p,g}$.

THEOREM 11. *With p, g as in Lemma 9, every matrix A in $M(J_p)$ has an eigenvalue in $J_{p,g}$. If A has no eigenvalue in J_p, or if A has at least one eigenvalue in $J_{p,g}$ which is not in J_p, then A has exactly* 2 (*distinct*) *eigenvalues in $J_{p,g}$.*

Consider the equation $x^2-tx+\Delta=0$. As in Lemma 10, this has solutions $x=(t+D^{1/2})/2$ where $D=t^2-4\Delta$. This is resolved into cases similar to standard discussions of quadratics. If $D=0$, A has only one eigenvalue $t/2$; if D is a nonzero square in J_p, A has 2 eigenvalues in J_p; and if D is a nonsquare, A has 2 eigenvalues in $J_{p,g}$ which are not in J_p. These eigenvalues are not equal since $D^{1/2}$ will have two values u and $-u$ where $u^2=D$, and $u=-u$ would force $D=0$.

THEOREM 12. *Let $A\in M(J_p)$ have eigenvalues α, β in $J_{p,g}$. Then A has a spectral decomposition* (1).

The proof is exactly the same as that in Section 8. Now the matrices E, F have coefficients in $J_{p,g}$.

11. Solution of the problem for all $A \in M(J_p)$, p prime, $p \neq 2$.

The only case not yet treated is that in which A has no eigenvalues in J_p. Choose a nonsquare n in J_p, (there surely exists one since the map $x \to x^2$ is 2 to 1), and let $g(v) = v^2 - n$. Then $A = \alpha E + \beta F$ as in Theorem 12. Now $J_{p,g}$ is a field with p^2 members. It follows that for each nonzero $h \in J_{p,g}$, $h^{p^2-1} = 1$. (One proof runs thus. Let $F = (0, a_1, a_2, \cdots, a_m)$ be a finite field, and $h \in F$, $h \neq 0$. The map $a_i \to ha_i$ is a permutation of F; hence $(ha_1)(ha_2) \cdots (ha_m) = a_1a_2 \cdots a_m$. Thus $h^m = 1$.) By the usual argument, the order of h is a divisor of $p^2 - 1$. The order of A is the least common multiple of the orders of α and β and thus can be no larger than $p^2 - 1$. That it can actually be as large as $p^2 - 1$ follows from the existence in $J_{p,g}$ of an element z of order $p^2 - 1$. (This is a special case of a theorem given in [4], p. 317, Theorem 7.B.) Then any matrix with z as an eigenvalue will have order $p^2 - 1$. To construct such a matrix we write $z = a + bv$ and find that $z^2 = a^2 + 2abv + b^2v^2 = a^2 + 2a(z-a) + b^2n = c + 2az$ where $c = b^2n - a^2$. Thus $z^2 - 2az - c = 0$. Let A be a matrix with trace $2a$ and determinant $-c$. Then the characteristic equation of A is $A^2 - 2aA - cI = 0$ and so z is an eigenvalue of A.

We may also put a lower bound on the order of A; namely, it can be no less than $p+1$. The reason for this is that if k is a divisor of $p-1$, the equation $x^k = 1$ has k solutions in J_p and can have no more in $J_{p,g}$. Thus, since the eigenvalues of A are not in J_p, they can satisfy no such equation so their order is a divisor of $p^2 - 1$ which is more than $p-1$.

12. Summary for $M(J_p)$, p prime. If A has two eigenvalues in J_p, the possible orders of A are the divisors of $p-1$. If A has one eigenvalue, the possible orders are kp where k is a divisor of $p-1$. If A has no eigenvalues, the possible orders are divisors of $p^2 - 1$ which are at least $p+1$.

13. Solution of the problem in $M(J_n)$, n squarefree. It may be noticed that in J_{15}, each member is uniquely determined by its values in J_3, J_5. For example $12 \in J_{15}$, and in J_3, $12 = 0$; in J_5, $12 = 2$. No other member of J_{15} leads to these two values, 0 and 2 in J_3, J_5, respectively. Thus we write $15 \equiv (0, 2)$. This leads to a one-to-one correspondence between J_{15} and $J_3 \oplus J_5$, where the latter symbol stands for all pairs (x, y), $x \in J_3$, $y \in J_5$. It is soon recognized that this correspondence is actually an isomorphism, that is, that addition and multiplication correspond. As an example, $12 \in J_{15}$ corresponds to 0 in J_3, 2 in J_5; 8 in J_{15} corresponds to 2 in J_3, 3 in J_5. Write $12 = (0, 2)$; $8 = (2, 3)$. Then $12 + 8 = (0+2, 2+3) = (2, 0) = 5$ and $12 \times 8 = (0 \times 2, 2 \times 3) = (0, 1) = 6$. A general result of this form is that if m and n are relatively prime, J_{mn} is isomorphic with $J_m \oplus J_n$, under the map $k \to (\alpha, \beta)$ where α and β are the forms taken by k in J_m and J_n respectively. (α is the remainder when k is divided by m.) Now if m and n are primes, and $x \in J_{mn}$ is idempotent, $x = (\alpha, \beta)$, then α and β are idempotents in the fields J_m and J_n respectively. Thus α and β are 0 or 1. Hence J_{mn} has 4 idempotents. Let $A \in M(J_{mn})$; then

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (\alpha_1, \alpha_2) & (\beta_1, \beta_2) \\ (r_1, r_2) & (\delta_1, \delta_2) \end{bmatrix}$$

which we may write in the form

$$A = B \oplus C \text{ with } B = \begin{bmatrix} \alpha_1 & \beta_1 \\ r_1 & \delta_1 \end{bmatrix} \in M(J_m),$$

and C equal to a similar expression. Clearly $A^k = B^k \oplus C^k$ and so the order of A is the least common multiple of the orders of B and C.

We omit the discussion of $M(J_n)$ where n has repeated prime factors because this theory is not yet in a satisfactory state.

14. An intractable problem. List all the matrices in $M(J_n)$ and opposite each, write its order. Let $s(n)$ be the sum of all these orders. Thus $s(n) = \sum u \cdot v$ where u is the number of matrices of order v. Then $s(2) = 26$, $s(3) = 272$, $s(5) = 4370$. What is $s(n)$ for other values of n?

15. Addendum. We have just learned that the spectral decomposition (5) may also be found in J. H. M. Wedderburn, Lectures on Matrices, Volume 17 of the American Mathematical Society Colloquium Publications. See pages 26, 27. See also J. H. Hodges, Amer. Math. Monthly, 73 (1966) 277–278.

I thank Professor G. E. Raynor for helpful discussions.

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LINE OF FLIGHT FROM SHOCK RECORDINGS

WALTER P. REID, U. S. Naval Ordnance Laboratory, Silver Spring, Maryland

Assume that a missile is traveling with constant speed in a straight line, and that it generates a shock wave that is conical, except perhaps in the neighborhood of the nose of the missile. Some devices record the times t at which the shock reaches them. In this paper equations will be developed for the speed and which we may write in the form

$$A = B \oplus C \text{ with } \quad B = \begin{bmatrix} \alpha_1 & \beta_1 \\ r_1 & \delta_1 \end{bmatrix} \in M(J_m),$$

and C equal to a similar expression. Clearly $A^k = B^k \oplus C^k$ and so the order of A is the least common multiple of the orders of B and C.

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Assume that a missile is traveling with constant speed in a straight line, and that it generates a shock wave that is conical, except perhaps in the neighborhood of the nose of the missile. Some devices record the times t at which the shock reaches them. In this paper equations will be developed for the speed and path of the missile and cone angle of the shock wave in terms of the times t. This is just a problem in analytic geometry. A simplified version, with ϕ or γ equal to zero, might be appropriate for classroom use. An alternate treatment, using a different arrangement of microphones, and a different calculation procedure, has been given by Zaroodny [1].

Let $x'y'z'$ be a fixed, orthogonal, right-handed set of axes, with recording devices on the y' axis at $y'=nL$, $n=-2, -1, 0, 1, 2, 3$. A second set of axes xyz with the same origin is to be oriented with the $-x$ axis in the direction of motion of the missile, and with the y axis in the $x'y'$ plane making an angle γ with the y' axis. Denote the angle between xy plane and $x'y'$ plane by ϕ. That is, the coordinates are to be related as follows:

$$(1)\qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\phi\cos\gamma & \cos\phi\sin\gamma & \sin\phi \\ -\sin\gamma & \cos\gamma & 0 \\ -\sin\phi\cos\gamma & -\sin\phi\sin\gamma & \cos\phi \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}.$$

The square matrix in (1) is orthogonal, so its inverse is its transpose. Thus (1) is easily solved for x', y', and z'. The relative orientations of the axes are shown in Figure 1.

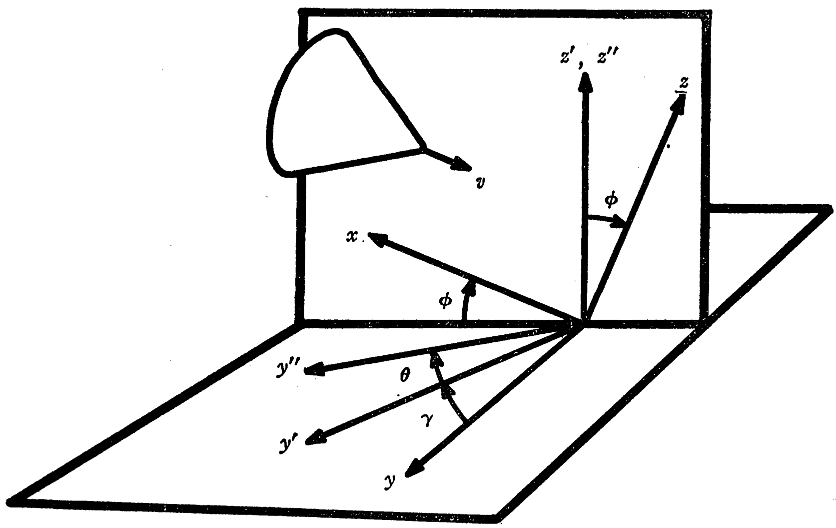

FIG. 1. Line of Flight from Shock Recordings.

Let the equation of the conical shock at time $t=0$ be

$$(2)\qquad a^2(x-h)^2 = (y-k)^2 + (z-b)^2$$

where a is the tangent of the cone angle. Since the shock is traveling with speed v in the $-x$ direction, it travels a distance vt_n before striking the nth recording device on the y' axis. The x, y, z coordinates of this device may be obtained from (1) by putting $(x', y', z')=(0, nL, 0)$. Hence one gets

(3) $a^2(vt_n + nL\cos\phi\sin\gamma - h)^2 = (nL\cos\gamma - k)^2 + (nL\sin\phi\sin\gamma + b)^2.$

Put

$$A_i = t_i^2 - 3t_{i-1}^2 + 3t_{i-2}^2 - t_{i-3}^2, \tag{4}$$

$$B_i = it_i - 3(i-1)t_{i-1} + 3(i-2)t_{i-2} - (i-3)t_{i-3}, \quad \text{and} \tag{5}$$

$$C_i = t_i - 3t_{i-1} + 3t_{i-2} - t_{i-3}. \tag{6}$$

Then one finds from (3) that

$$\frac{v}{2h} = \frac{C_2B_1 - C_1B_2}{A_2B_1 - A_1B_2} = \frac{C_3B_2 - C_2B_3}{A_3B_2 - A_2B_3} \quad \text{and} \tag{7}$$

$$\frac{v}{2L\cos\phi\sin\gamma} = \frac{B_1C_2 - B_2C_1}{A_2C_1 - A_1C_2}. \tag{8}$$

From (7) one obtains a relationship between the t's. Thus the t's are dependent, and one of them, say t_3, can be calculated from the others. For this reason, t_3 will subsequently not be included in the equations. Next put

$$vt_n + nL\cos\phi\sin\gamma - h = vT_n, \tag{9}$$

$$2D = T_1^2 - 2T_0^2 + T_{-1}^2, \qquad 4E = T_1^2 - T_{-1}^2, \tag{10}$$

and note that the T's are now known. So from (3), (9), and (10) one gets

$$(vaT_0)^2 = k^2 + b^2, \tag{11}$$

$$v^2a^2E = bL\sin\phi\sin\gamma - kL\cos\gamma, \quad \text{and} \tag{12}$$

$$v^2a^2D = (L\sin\phi\sin\gamma)^2 + (L\cos\gamma)^2. \tag{13}$$

As things now stand, there are five recording devices, and the five equations: (7), (8), (11), (12), (13). More equations are needed, and hence more shock recording stations. However, the instrument at $(x', y', z') = (0, 3L, 0)$ was found not to give added information. Hence additional positions for recording shocks will not be selected on the y' axis, but instead on a y'' axis whose orientation will be prescribed next.

Rotate the $x'y'z'$ axes by an angle θ about the z' axis to form a new set of coordinate axes $x''y''z''$, and place recording devices on the y'' axis at $y'' = nS$, with $n = -2, -1, 1, 2$. There is already an instrument at $y'' = 0$, since this coincides with $y' = 0$ on the y' axis. The angle ϕ used before will not be changed because the $x'y'$ axes and the $x''y''$ axes are in the same plane. However, the angle γ will be replaced by some other angle λ, where

$$\lambda = \gamma + \theta. \tag{14}$$

The calculations using the new recording stations will be based on the same equations as above, but with a different notation. In (1)–(13), make the following changes: $\gamma \to \lambda$, $(x', y', z') \to (x'', y'', z'')$, $t \to \tau$, $A \to F$, $B \to G$, $C \to J$, $D \to M$, $E \to Q$, $L \to S$ and $T \to \Upsilon$, and note that $t_0 = \tau_0$ and $T_0 = \Upsilon_0$. The new equations obtained are then:

$$\text{(15)} \qquad \frac{v}{2h} = \frac{J_2G_1 - J_1G_2}{F_2G_1 - F_1G_2},$$

$$\text{(16)} \qquad \frac{v}{2S \cos \phi \sin \lambda} = \frac{G_1J_2 - G_2J_1}{F_2J_1 - F_1J_2},$$

$$\text{(17)} \qquad v^2a^2Q = bS \sin \phi \sin \lambda - kS \cos \lambda, \quad \text{and}$$

$$\text{(18)} \qquad v^2a^2M = (S \sin \phi \sin \lambda)^2 + (S \cos \lambda)^2.$$

When $v/2h$ is eliminated from (7) and (15), an equation connecting the t's and τ's is obtained. Thus there is at least one more recording station than is needed. However, for convenience the extra station will be kept.

From equations (8), (11)–(14), and (16)–(18) the following relations are easily determined:

$$\text{(19)} \qquad \frac{\sin \gamma}{\sin(\gamma + \theta)} = \frac{(A_2C_1 - A_1C_2)(G_1J_2 - G_2J_1)\, S}{(B_1C_2 - B_2C_1)(F_2J_1 - F_1J_2)\, L},$$

$$\text{(20)} \qquad k = \frac{LS \sin(\lambda + \gamma)(LQ \sin \gamma - SE \sin \lambda)}{S^2D \sin^2 \lambda - L^2M \sin^2 \gamma},$$

$$\text{(21)} \qquad \sin^2 \phi = \frac{S^2D \cos^2 \lambda - L^2M \cos^2 \gamma}{L^2M \sin^2 \gamma - S^2D \sin^2 \lambda}, \quad \text{and}$$

$$\text{(22)} \qquad b^2 = \frac{\Upsilon_0^2 L^2 S^2 \sin(\lambda + \gamma) \sin \theta}{S^2D \sin^2 \lambda - L^2M \sin^2 \gamma} - k^2.$$

Now calculate the unknown parameters as follows: γ from (19), λ from (14), k from (20), ϕ from (21), b from (22), v from (8) or (16), h from (7) or (15), and a from (11). The sign of $b \sin \phi$ can be established from (12) or (17), but the signs of b and ϕ are not known. This is because the arrangement of recording devices cannot distinguish between a moving object and its mirror image on the other side of the $y'y''$ plane. However, this is presumably known. If not, an additional shock recording instrument not in the $y'y''$ plane can be used.

The calculated values of the parameters can be checked by substituting them into (3) and (14), and the counterpart of (3) with t replaced by τ and γ replaced by λ, for $n = -2, -1, 0, 1$ and 2.

It is now possible to ascertain past and future locations of the moving object. As an example, it will first be found when and where the vertex of the shock cone will strike the plane $x' = -R$. Identify this point by means of the subscript α.

It is evident from (2) that the missile is traveling along the line $(y, z) = (k, b)$. So the point α has coordinates $(x_\alpha', y_\alpha, z_\alpha) = (-R, k, b)$. From (1), therefore, it is found that

$$\text{(23)} \qquad x_\alpha = \frac{k \sin \gamma + b \sin \phi \cos \gamma - R}{\cos \phi \cos \gamma},$$

$$\text{(24)} \qquad y_\alpha' = \frac{k - R \sin \gamma}{\cos \gamma}, \quad \text{and}$$

(25) $$z_\alpha' = \frac{k \sin \gamma \sin \phi + b \cos \gamma - R \sin \phi}{\cos \phi \cos \gamma}.$$

The y' and z' coordinates of the point where the missile will pass through the plane $x' = -R$ are given by (24) and (25). The time at which the vertex of the cone will strike the plane is $(h - x_\alpha)/v$.

As a second application, let it be assumed that two missiles have come from the same source, and that the location of the source is desired. In theory, one need only find the point of intersection of the two calculated lines of flight. Actually, however, the chance that two computed flight paths will intersect is rather remote. There will be inaccuracies in measurement, and perhaps even some motion of the source. Also, the trajectories are actually more nearly parabolic than straight. So the procedure used will be to determine the intersection of two vertical planes, each containing a line of flight.

Assume that gravity acts in the z' direction. Determine for each path the projection plane parallel to the z' axis. Call this the "plane of flight." Then even if the path is actually concave downward, due to gravity, rather than straight as calculated, it will still lie in this plane of flight. Moreover, two nonparallel planes of flight will always intersect. This will give the x' and y' coordinates of the source, which should be useful, and perhaps sufficient.

For two different missiles, one has

(26) $$x' \sin \gamma_1 - y' \cos \gamma_1 = -k_1,$$

(27) $$x' \sin \gamma_2 - y' \cos \gamma_2 = -k_2.$$

Equations (26) and (27) may be solved simultaneously to give the point (x', y') at which the planes of flight intersect, and hence the presumed location of the source.

Reference

1. S. J. Zaroodny, Trajectory indicator—a proposal, SIAM J. Appl. Math., 14 (1966) 1366–1389

SAM FUNCTIONS

WAYNE HALL AND DONALD W. HIGHT, Kansas State College

A SAM function f is defined to be a continuous, subadditive, and multiplicative real valued function on a subset H of the set R of real numbers; that is,

$$f \text{ is continuous on } H,$$

$$f(x+y) \leqq f(x) + f(y) \ \forall x, y \in H,$$

and

$$f(xy) = f(x)f(y) \ \forall x, y \in H.$$

$$z'_\alpha = \frac{k \sin \gamma \sin \phi + b \cos \gamma - R \sin \phi}{\cos \phi \cos \gamma}. \tag{25}$$

The y' and z' coordinates of the point where the missile will pass through the plane $x' = -R$ are given by (24) and (25). The time at which the vertex of the cone will strike the plane is $(h - x_\alpha)/v$.

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and

$$f(xy) = f(x)f(y) \ \forall x, y \in H.$$

For the purposes of this paper, we shall consider H to be the set R of real numbers, the set I of integers, or one of the rays $(-\infty, 0)$, $(-\infty, 0]$, $(0, \infty)$, or $[0, \infty)$.

Obviously, the absolute value function and the constant functions $f(x)=0$ and $f(x)=1$ are SAM functions on any subset of R. Other characteristics of SAM functions follow:

LEMMA 1. *If f is a* SAM *function on a set H and $0\in H$, then* $f(x)=1$ or $f(0)=0$.

Proof. Since $f(0)=f(x\cdot 0)=f(x)f(0)$, it follows that $0=f(0)(f(x)-1)$. Therefore, $f(x)=1$ or $f(0)=0$.

LEMMA 2. *If f is a* SAM *function on a set H, $1\in H$, and $f(1)=0$, then $f(x)=0$ on H.*

Proof. Since $f(x)=f(1\cdot x)=f(1)f(x)$, and $f(1)=0$, then $f(x)=0$.

LEMMA 3. *If f is a* SAM *function on a set H and $1\in H$, then* $f(x)=0$ or $f(1)=1$.

Proof. Since $f(1)=f(1\cdot 1)=f(1)f(1)$, it follows that $0=f(1)(f(1)-1)$. Therefore, $f(1)=0$ or $f(1)=1$. Then, from Lemma 2, $f(x)=0$ or $f(1)=1$.

If $f(x)=0$ or $f(x)=1$ on a set H, we shall call f a *trivial* SAM function. Otherwise, f is *nontrivial*. Thus the above lemmas substantiate the following theorem:

THEOREM 1. *If f is a nontrivial* SAM *function on a set H that contains* 0 *and* 1' *then $f(0)=0$ and $f(1)=1$.*

LEMMA 4. *If f is a nontrivial* SAM *function on a set H that contains* -1 *and* 1, *then either $f(-1)=1$ or $f(-1)=-1$.*

Proof. Since $f(1)=f(-1\cdot -1)=f(-1)f(-1)$, it follows that $1=f^2(-1)$ and therefore $f(-1)=-1$ or $f(-1)=1$.

THEOREM 2. *If f is a nontrivial* SAM *function on a set H that contains* 0, 1 *and* -1 *then f is either an odd function or an even function.*

Proof. Since f is a SAM function and H contains 0, 1, and -1, it follows that $f(0)=0$, $f(1)=1$, and $f(-1)=-1$ or $f(-1)=1$.

Case I. $f(-1)=-1$. Since $f(-1)=-1$, then $f(-x)=f(-1)f(x)=-f(x)$. Thus, f is an odd function

Case II. $f(-1)=1$. Since $f(-1)=1$, then $f(-x)=f(-1)f(x)=f(x)$. Thus f is an even function. Therefore, if f is a nontrivial SAM function on H, it is either an odd function or an even function.

Every measurable (continuous) odd subadditive function is of the type $f(x)=mx$, m a constant [1]. Thus odd SAM functions are characterized by determining permissible values of m.

THEOREM 3. *The only nontrivial odd* SAM *function has the equation $f(x)=x$.*

Proof. Let f be an odd subadditive function on a set H. By the above result, $f(x)=mx$, m a constant. Thus for $xy\neq 0$, $f(xy)=mxy$ and $f(x)f(y)=m^2xy$. If f is to be a SAM function, then $f(xy)=f(x)f(y)$ and $mxy=m^2xy$. Therefore $m=0$ or

$m=1$. Thus if f is a nontrivial odd SAM function, $f(x)=x$.

Since each nontrivial SAM function on a set H that contains 1, 0, and -1 has been proved to be either an odd function or an even function and all nontrivial odd SAM functions have been shown to be the function $f(x)=x$, the nontrivial even SAM functions remain to be characterized.

Laatsch [**3**] proved a theorem which completely characterizes even subadditive functions. The theorem is stated below as Lemma 5.

LEMMA 5. *Let f be subadditive on $[0, \infty)$. Then f can be extended to an even subadditive function F on R if and only if $f(x-y) \leqq f(x)+f(y)$ for all $x \geqq y$ in $[0, \infty)$.*

The next theorem is a summarization of Theorems 1 and 2 and Lemma 5.

THEOREM 4. *If f is a nontrivial even* SAM *function on a set H containing* 1, 0 *and* -1, *then* $f(1)=1$, $f(0)=0$, $f(-1)=1$ *and* $f(x-y) \leqq f(x)+f(y)$ *for* $x \geqq y$ *in* $[0, \infty)$.

Since any nontrivial even SAM function must have the special properties described in Theorem 4 it would seem natural to ask the following questions:

1. Is there any SAM function other than the absolute value function that satisfies the conditions of Theorem 4?
2. Are all functions (if there are any) that satisfy the conditions of Theorem 4 SAM functions?

These questions are answered in the following Lemmas.

LEMMA 6. *There exist infinitely many nontrivial even* SAM *functions on R.*

Proof. Note that $f_n(x)=\sqrt[n]{x}$, n a natural number is a continuous monotone increasing function on $[0, \infty)$. Thus $f_n(x-y)=\sqrt[n]{x-y} \leqq \sqrt[n]{x}$ for $x \geqq y$. But $\sqrt[n]{y} \geqq 0$ on $[0, \infty)$. Therefore, $f_n(x-y) \leqq f_n(x)+f_n(y)$ for $x \geqq y$ in $[0, \infty)$. Since $f_n(x)=\sqrt[n]{x}$, n a natural number, is subadditive and has the property $f(x-y) \leqq f(x)+f(y)$ for all $x \geqq y$ in $[0, \infty)$, then, by Lemma 5, f_n can be extended to an even function F_n on R. Hence we define $F_n(x)=f_n(|x|)=\sqrt[n]{|x|}$. Thus F_n is subadditive and continuous on R. Also,

$$F_n(xy) = \sqrt[n]{|xy|} = \sqrt[n]{|x| \cdot |y|} = \sqrt[n]{|x|} \cdot \sqrt[n]{|y|} = F_n(x)F_n(y);$$

thus F_n is multiplicative on R. Therefore F_n is a nontrivial SAM function on R that satisfies the conditions of Theorem 4, and since n is a natural number, there exist infinitely many such SAM functions on R.

LEMMA 7. *There exists an even subadditive function that satisfies the conditions of Theorem* 4, *but it is not a* SAM *function.*

Proof. Existence is verified by an example; let $G(x)=|\sin(\pi x/2)|$. Subadditivity and the condition $G(x-y) \leqq G(x)+G(y)$ follow from elementary trigonometric identities. Also, $G(0)=0$, $G(1)=1$, and $G(-1)=1$. However, G is not multiplicative on a set H since

$$G(2/1 \times 4/3) = |\sin(\pi/2 \times 2/1 \times 4/3)| = \sqrt{3}/2 \quad \text{but}$$
$$G(2/1)G(4/3) = |\sin \pi||\sin(2\pi/3)| = 0.$$

Thus, with the above lemmas and theorems, we have shown that every nontrivial SAM function on a set H containing 0, 1, and -1 must be $f(x)=x$ or it must be an even function. Then if the function is an even function it must satisfy these properties: (i) $f(0)=0$, $f(1)=1$, $f(-1)=1$; (ii) $f(x)=f(|x|)$ (it is even); (iii) $f(x-y)\leqq f(x)+f(y)$ for all $x\geqq y$ in $[0, \infty)$; and (iv) although not all nontrivial SAM functions satisfy these properties, infinitely many do. However, these conditions are necessary but not sufficient since at least one function that is not multiplicative satisfies (i) through (iv).

W. Hall is presently an NDEA fellow at George Peabody College for Teachers. D. Hight is presently an NSF Science Faculty Fellow at the University of Massachusetts.

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1. R. Cooper, The converse of the Cauchy-Hölder inequality and the solutions of the inequality $g(x+y)\leqq g(x)+g(y)$, Proc. London Math. Soc., 2, 26 (1927) 415–432.

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3. R. G. Laatsch, Subadditive functions of one real variable, unpublished Ph.D. dissertation, Oklahoma State University, 1962.

A NEW APPROACH TO CIRCULAR FUNCTIONS, II AND *lim* $(\sin x)/x$

GERSON B. ROBISON, SUNY at New Paltz

In the September issue of this MAGAZINE, W. F. Eberlein gives a nongeometric definition of the circular functions by way of the definite integral [1]. There is another elementary nongeometric approach which seems not to be generally known, that by way of the functional equation.

THEOREM I. *Let p be a positive real number. Let S and C be real valued functions of a real variable satisfying:*

A. $C(x-y)=C(x)C(y)+S(x)S(y)$ for all x, y.

B. $S(p)=1$

C. $S(x)\geqq 0$ *for all* $x\in[0, p]$.

Then S and C are uniquely determined.

With $p=\pi/2$, S and C are thus the sine and cosine functions respectively. We establish the theorem with a sequence of lemmas, most of whose proofs are quite trivial, and have been omitted.

LEMMA 1. $C(0)=1$.

Proof. $C(0)=C(p-p)=(C(p))^2+(S(p))^2=(C(p))^2+1$. Hence $C(0)\geqq 1$. Also $C(0)=C(0-0)=(C(0))^2+(S(0))^2\geqq(C(0))^2$. By the first part, $C(0)>0$, so we may divide to get $1\geqq C(0)$. Hence $C(0)=1$.

LEMMA 2. $C(-x)=C(x)$. *Proof:* Use $C(-x)=C(0-x)$.

Thus, with the above lemmas and theorems, we have shown that every nontrivial SAM function on a set H containing 0, 1, and -1 must be $f(x)=x$ or it must be an even function. Then if the function is an even function it must satisfy these properties: (i) $f(0)=0$, $f(1)=1$, $f(-1)=1$; (ii) $f(x)=f(|x|)$ (it is even); (iii) $f(x-y)\leqq f(x)+f(y)$ for all $x\geqq y$ in $[0, \infty)$; and (iv) although not all nontrivial SAM functions satisfy these properties, infinitely many do. However, these conditions are necessary but not sufficient since at least one function that is not multiplicative satisfies (i) through (iv).

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LEMMA 2. $C(-x)=C(x)$. *Proof:* Use $C(-x)=C(0-x)$.

LEMMA 3. (a) $(C(x))^2+(S(x))^2=1$. *Proof.* Use $C(x-x)=C(0)$.
(b) $0\leqq S(x)\leqq 1$ for all $x\in[0, p]$.

LEMMA 4. $S(0)=C(p)=0$.

LEMMA 5. (a) $C(p-x)=S(x)$. (b) $S(p-x)=C(x)$. *Proof.* Use 5a.
(c) $S(p/2)=C(p/2)$. (d) $0\leqq C(x)\leqq 1$ for all $x\in[0, p]$.

LEMMA 6. (a) $S(x+y)=S(x)C(y)+C(x)S(y)$. *Proof.* Use $S(x+y)=C((p-x)-y)$.
(b) $S(2x)=2S(x)C(x)$. (c) $S(2p)=0$.

LEMMA 7. (a). $2(C(p/2))^2=2(S(p/2))^2=1$. (b) $S(p/2)=C(p/2)=\sqrt{2}/2$.

LEMMA 8. $S(-p/2)=-S(p/2)=-\sqrt{2}/2$. *Proof.* Use $S(p/2+(-p/2))=0$.

LEMMA 9. $S(-p)=-1$.

LEMMA 10. $C(2p)=-1$.

LEMMA 11. $S(-x)=-S(x)$. *Proof.* Use $S(-x)=C(p-(-x))=C(-p-x)$.

LEMMA 12. (a) $C(x+y)=C(x)C(y)-S(x)S(y)$. (b) $C(2x)=(C(x))^2-(S(x))^2=2(C(x))^2-1$. (c) $(C(x)+1)/2=(C(x/2))^2$,

LEMMA 13. $S(x-y)=S(x)C(y)-C(x)S(y)$.

LEMMA 14. (a) $S(p+x)=C(x)$. (b) $C(p+x)=-S(x)$. (c) $S(2p+x)=-S(x)$. (d) $C(2p+x)=-C(x)$. (e) $S(4p+x)=S(x)$. (f) $C(4p+x)=C(x)$.

Lemmas 2, 11, and 14 show that S and C are uniquely defined provided that they are so defined for $[0, p]$. In particular, 14e and 14f exhibit the periodicity. Our remaining task is to show that the functions are unique for the closed interval. In the remaining lemmas x is to be assumed restricted to $[0, p]$, and so the corresponding ranges of C and S lie within $[0, 1]$.

LEMMA 15. (a) *C is nonincreasing on $[0, p]$.* *Proof.* Apply 3b and 5d to 12a to show that $C(x+y)\leqq C(x)$.
(b) *S is nondecreasing on $[0, p]$.* *Proof.* Use 5a and 15a.

LEMMA 16. *For any constant c, if $C(x)\geqq 1-c$, then $C(x/2)\geqq 1-c/2$.*

Proof. $C(x/2)\geqq(C(x/2))^2=(1+C(x))/2\geqq(1+(1-c))/2=1-c/2$.

LEMMA 17. $\lim_{n\to\infty} C(p/2^n)=1$

Proof. Since $C(p)=0\geqq 1-1$, we can use induction on Lemma 16 to get $C(p/2^n)\geqq 1-1/2^n$ for every positive integer n. But $C(x)\leqq 1$ for all x.

LEMMA 18. (a) *C is continuous at zero.* *Proof.* 17 and 15a.
(b) *S is continuous at zero.*

LEMMA 19. *S and C are continuous.* *Proof.* 18, 6a and 12a, with $y=\Delta x$.

LEMMA 20. *S and C are uniquely defined on $[0, p]$.*

Proof. If $C(x)$ is known, induction on 12a gives us $C(kx)$ and induction on 12c gives us $C(x/2^n)$ for any integers k and n. Since we do know $C(p)=0$, we can get $C(x)$ for any $x=kp/2^n$, $x\in[0, p]$. Such values of x are dense in the interval, so the lemma follows by continuity of C, and 5a.

We leave the (independent) proof of the following theorem to the interested reader. Its proof is a little shorter than that of Theorem I.

THEOREM II. *Let q be a positive real number. Let T be a real-valued function defined for all real x not of the form $(2+4n)q$, n any integer. Let T satisfy the following conditions:*

A. $[1+T(x)T(y)]T(x-y)=T(x)-T(y)$

B. $T(q)=1$

C. $T(x)\geqq 0$ *for all* $x\in[0, q]$.

Then, T is uniquely determined.

The following objection might now be raised: it is true that the usual sine and cosine functions are uniquely determined by A, B, and C of Theorem I if p is taken as $\pi/2$, but how do we define the number π?

From the standpoint of the elementary geometric applications of the circular functions, this question is irrelevant. One only has to take an arbitrary p, say $p=1$ or $p=90$, as the measure of a right angle. The magic of radian measure lies in the role that the circular functions play in the calculus. The key to the peculiar virtues of radian measure in that context can be found in the fact that $\lim_{x\to 0}(\sin x)/x=1$. The following theorems will show that we can define π and obtain this limit without recourse to circles or integral theory.

For the remainder of this paper, S and C will represent the functions defined in Theorem I with $p=1$. We shall restrict the domain of both functions to $[0, 1]$. F will be the function defined by: $F(x)=(\sin x)/x$, $0<x\leqq 1$. It is continuous on its domain.

THEOREM III. *$F(x)$ is a decreasing function on* $(0, 1]$.

LEMMA. *For any x and positive integer n such that $0<x<nx\leqq 1$, the sequence $F(x)$, $F(2x)$, $\cdots$, $F(nx)$ decreases.*

Let $n=2$.

$$F(2x)=\frac{S(2x)}{2x}=\frac{2S(x)C(x)}{2x}=[F(x)]C(x)<F(x).$$

Assume: $F(kx)<F((k-1)x)$. To show: $F((k+1)x)<F(kx)$. From our assumption, after simplifying, we get $kS(kx)-S(kx)<kS(kx)C(x)-kC(kx)S(x)$. But

$$\begin{aligned} kS(kx)C(x)-S(kx) &\leqq kS(kx)-S(kx) \text{ (Lemma 5d, Theorem I)}\\ &< kS(kx)C(x)-kC(kx)S(x) \text{ (by the above)}\\ &\leqq kS(kx)-kC(kx)S(x). \end{aligned}$$

Hence $kS(kx)C(x)+kC(kx)S(x)<kS(kx)+S(kx)$, i.e., $kS((k+1)x)<(k+1)S(kx)$. Dividing by $k(k+1)x$, we get $F(k+1)x<F(kx)$

Proof of theorem. Let $0<x_0<y\leqq 1$. Take k so that $2^{-k}x_0<y-x_0$. Let $x_1=x_0+2^{-k}x_0<y$. Then $x_0=2^k(2^{-k}x_0)$ and $x_1=(2^k+1)(2^{-k}x_0)$. Hence, by the above lemma, $F(x_0)>F(x_1)$. Let $\epsilon=F(x_0)-F(x_1)>0$. Since F is continuous, there is a $\delta>0$ such that for $y-\delta<t<y$ we have $|F(y)-F(t)|<\epsilon$. In particular, if $y-\delta<x_1<y$, then $F(y)-F(x_1)\leqq|F(y)-F(x_1)|<\epsilon=F(x_0)-F(x)_1$, so that $F(y)<F(x_0)$. On the other hand, if $x_1\leqq y-\delta$, take the integer $j>0$ so that $2^{-(k+j)}x_0<\delta$. Then for some positive integer n, $y-\delta<x_1+n(2^{-(k+j)})x_0<y$, by the archimedean property. Let $x_2=x_1+n(2^{-(k+j)})x_0$. Then $x_1=(2^j+2^{j+k})(2^{-(j+k)}x_0)$ and $x_2=(2^j+2^{j+k}+n)(2^{-(j+k)}x_0)$.

Again by the lemma, $F(x_2)<F(x_1)$, and

$$F(y)-F(x_2)\leqq|F(y)-F(x_2)|<\epsilon=F(x_0)-F(x_1)<F(x_0)-F(x_2).$$

So again, $F(y)<F(x_0)$.

THEOREM IV. *$\operatorname{Lim}_{x\to 0} F(x)$ exists.*

LEMMA 1. *For any nonnegative integer n*

$$\frac{2}{1+C(2^{-n})}\leqq\frac{2}{2-2^{-n}}. \tag{1}$$

Proof. By Lemma 16 of Theorem I, if $C(2^{-n})\geqq 1-2^{-n}$, then $C(2^{-(n+1)})\geqq 1-2^{-(n+1)}$. Since $C(2^{-0})=C(1)=0\geqq 1-2^{-0}$, induction gives us $C(2^{-n})\geqq 1-2^{-n}$ for every nonnegative integer n. Hence (1) holds.

LEMMA 2. *The sequence $\{F(2^{-i})\}$, $i=1, 2, \cdots$ has a limit.*

We shall use induction on $F^2(2^{-i})$ $(=[F(2^{-i})]^2)$ to show that the sequence is bounded. Specifically, we show that $F^2(2^{-n})\leqq(2^{n+1})/(2^{n-1}+1)<4$. By Theorem III, the sequence is increasing, so it will therefore have a limit. We shall use the fact, easily verified by using the properties developed in the proof of Theorem I, that $F^2(x/2)=2/(1+C(x))F^2(x)$.

Proof. $F^2(2^{-1})=2/(1+C(1))F^2(1)=2\leqq(2^{1+1})/(2^{1-1}+1)$.
Assume

$$F^2(2^{-k})\leqq\frac{2^{k+1}}{2^{k-1}+1},$$

to show

$$F^2(2^{-(k+1)})\leqq\frac{2^{k+2}}{2^k+1} \tag{2}$$

$$\begin{aligned}F^2(2^{-(k+1)})&=\frac{2}{1-C(2^{-k})}F^2(2^{-k})\\&\leqq\frac{2}{2-2^{-k}}\cdot\frac{2^{k+1}}{2^{k+1}+1}\\&=\frac{2^{k+2}}{2^k+1+(1-2^{-1}-2^{-k})}.\end{aligned}$$

For $k \geqq 1$, the expression in the parenthesis is nonnegative, so (2) holds.

Proof of theorem. Lemma 2 and Theorem III.

DEFINITION. $\pi = 2 \lim_{x\to 0} F(x)$.

DEFINITION. Sin $x = S(2x/\pi)$, cos $x = C(2x/\pi)$.

It is easily verified that these functions satisfy the conditions A, B, and C with $p = \pi/2$.

THEOREM V. $\text{Lim}_{x\to 0} (\sin x)/x = 1$.

Proof. $1 = (2/\pi)(\pi/2) = (2/\pi)\lim_{x\to 0} S(x)/x = (2/\pi)\lim_{x\to 0} (S(2x/\pi))/(2x/\pi)$ $= \lim_{x\to 0} (S(2x/\pi)/x = \lim_{x\to 0} (\sin x)/x$.

Looking at this result from a more usual geometric approach, suppose that we had started with the assumption that a measure exists for directed angles, which is additive for adjacent angles and preserved under congruence. We could define S and C within the usual cartesian framework, taking the right angle as having unit measure. From this all the rest follows, with no need of arc length, area, integration, or infinite sums.

Reference

1. W. F. Eberlein, The circular function(s), this MAGAZINE, 39 (1966) 197–201.

ON $\sigma(xy) = \sigma(yx)$

SVETOZAR KUREPA, Georgetown University

1. If x and y are square matrices of the same finite order, then the set of eigenvalues of xy is the same as that of yx. This is usually proved by use of determinants or by use of a system of linear equations or by use of elementary transformations on matrices x, y. In this note we give another proof of this result by considering matrices as elements of the algebra of all matrices.

Let $\mathfrak{A}$ be an associative algebra with an identity e. An element $x \in \mathfrak{A}$ is said to be *right regular* if $xy = e$ holds for some $y \in \mathfrak{A}$; $v \in \mathfrak{A}$ is said to be *left regular* if $uv = e$ for some $u \in \mathfrak{A}$. If $x \in \mathfrak{A}$ is both left and right regular, then a unique element $x^{-1} \in \mathfrak{A}$ exists such that $xx^{-1} = x^{-1}x = e$. In this case we say that x is *regular*. The set G of all regular elements of $\mathfrak{A}$ is a group. Indeed, $e \in G$, and if $x \in G$, then x^{-1} is also regular. In fact, $(x^{-1})^{-1} = x$. Furthermore, if x, $y \in G$, then $y^{-1}x^{-1}(xy)$ $= y^{-1}(x^{-1}x)y = e$. Hence x, $y \in G$ implies $xy \in G$ and $(xy)^{-1} = y^{-1}x^{-1}$. From this it follows that G is a group.

PROPOSITION 1. *If x, $y \in \mathfrak{A}$ and at least two elements of the set $\{x, y, xy, yx\}$ are regular, then all of them are regular.*

Proof. We know that x, $y \in G \Rightarrow xy$, $yx \in G$. Now if x, $xy \in G$, then $y = x^{-1}(xy)$, as a product of regular elements, is in G. In the same way x, $yx \in G \Rightarrow y \in G$. Now

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PROPOSITION 2. *If $\mathfrak{A}$ is a finite dimensional algebra, then $xy\in G\Rightarrow x, y\in G$.*

Proof. For $z\in\mathfrak{A}$, the sequence $e, z, z^2, \cdots$ is dependent. There is, therefore, a smallest number m, depending on z, such that

$$(1)\qquad z^m = \nu_1 z^{m-1} + \cdots + \nu_{m-1}z + \nu_m e$$

which is called the *minimal equation* of z. From (1) we get

$$(2)\qquad z(z^{m-1} - \nu_1 z^{m-2} - \cdots - \nu_{m-1}e) = \nu_m e.$$

From (2) we conclude that z is regular if and only if $\nu_m\neq 0$. In this case

$$(3)\qquad z^{-1} = \frac{1}{\nu_m}(z^{m-1} - \nu_1 z^{m-2} - \cdots - \nu_{m-1}e).$$

Now we turn to the proof of Proposition 2. We will prove that $y\notin G$ implies $yx\notin G$ for every $x\in\mathfrak{A}$. Indeed, from $y\notin G$ it follows that the minimal equation of y is of the form

$$(4)\qquad y^s = \mu_1 y^{s-1} + \cdots + \mu_{s-1}y.$$

Let

$$(5)\qquad (yx)^K = \alpha_1(yx)^{K-1} + \cdots + \alpha_K e$$

be the minimal equation of yx. By use of $(yx)^n=y(xy)^{n-1}x$ and (5) we get

$$(6)\qquad yw = \alpha_K e, \quad w = [(xy)^{K-1} - \alpha_1(xy)^{K-2} - \cdots - \alpha_{K-1}e]x.$$

If we post-multiply (4) by w and if we use (6) we find that

$$(7)\qquad \alpha_K(y^{s-1} - \mu_1 y^{s-2} - \cdots - \mu_{s-1}e) = 0$$

which, together with the assumption that (4) is the minimal equation of y, implies $\alpha_K=0$. Since $\alpha_K=0$, we find that yx is not regular, i.e., $yx\notin G$. Thus

$$(8)\qquad y\notin G\Rightarrow yx\notin G \qquad (x\in\mathfrak{A}).$$

By Proposition 1 we conclude that at least one of the elements x and xy is not regular. The assumption $x\notin G$ (by the just proved property (8)) implies $xy\notin G$. Hence in any case $xy\notin G$. Thus $y\notin G\Rightarrow xy$, $yx\notin G$. This proves that $uv\in G$ can hold if and only if u and v are regular elements of $\mathfrak{A}$.

REMARK 1. Let $\mathfrak{A}$ be the algebra of all matrices of order two. Set

$$x = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \qquad y = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}.$$

Then $xy=y$, $yx=0$. Thus minimal equations of xy and yx are, in general, different. However, if x and y are regular elements of an associative finite dimensional algebra $\mathfrak{A}$, then xy and yx have the same minimal equation. Indeed, if we post-multiply (5) by y and premultiply by x, we get $[(xy)^K-\alpha_1(xy)^{K-1}- \cdots -\alpha_K e]xy=0$. Since xy is regular it follows that $(xy)^K=\alpha_1(xy)^{K-1}+ \cdots +\alpha_K e$, i.e., xy and yx have the same minimal equation.

Now, suppose that the minimal equation of xy is of order K, that the minimal equation of yx is of order m, and that $xy\notin G$. From $(xy)^K=\alpha_1(xy)^{K-1}+ \cdots +\alpha_{K-1}xy$ we get $(yx)^{K+1}=\alpha_1(yx)^K+ \cdots +\alpha_{K-1}(yx)^2$ which implies that $K+1\geqq m$. Hence, if $K\leqq m$, we have two possibilities:

$$\text{(a) } K+1=m, \qquad \text{(b) } K=m.$$

If $K=m$ and P and Q are the corresponding minimal polynomials of xy and yx respectively, then

$$yxP(yx)=0\Rightarrow \lambda P(\lambda)=Q(\lambda)(\lambda-\gamma)$$

and, in the same way,

$$\lambda Q(\lambda)=P(\lambda)(\lambda-\beta) \text{ with complex numbers } \beta \text{ and } \gamma.$$

From this we get $\lambda^2 P(\lambda)Q(\lambda)\equiv P(\lambda)Q(\lambda)(\lambda-\beta)(\lambda-\gamma)$ which implies that $\beta=\gamma=0$, and, therefore, that $P(\lambda)\equiv Q(\lambda)$. Hence if the orders of the minimal polynomials of xy and yx are the same, these polynomials are the same.

REMARK 2. The condition dim $\mathfrak{A}<\infty$ in Proposition 2 is essential, as is proved by the following example.

Let X be the vector space of all polynomials in one variable t with pointwise operations among polynomials as algebraic operations and let $\mathfrak{A}$ be the set of all linear operators from X into X. With the usual algebraic structure on $\mathfrak{A}$, $\mathfrak{A}$ is an algebra with an identity e.

Now we define two operators D and F as follows.

$$D(\alpha_0+\alpha_1 t+\cdots+\alpha_n t^n)=\alpha_1+2\alpha_2 t+\cdots+n\alpha_n t^{n-1} \quad \text{and}$$

$$F(\alpha_0+\alpha_1 t+\cdots+\alpha_n t^n)=\alpha_0 t+\frac{\alpha_1}{2}t^2+\cdots+\frac{\alpha_n}{n+1}t^{n+1},$$

Then, D, $F\in\mathfrak{A}$ and $DF=e$, while $FD1=0$, i.e., $FD\neq e$. Hence the product DF is regular while the product FD is not regular. In fact,

$$DF=e\Rightarrow D(FD-e)=0, \qquad (FD-e)F=0.$$

Thus D and F are divisors of zero in $\mathfrak{A}$.

REMARK 3. Let X be a separable Hilbert space, $e_0, e_1, \cdots$ an orthonormal basic set in X, and $L(X)$ the Banach algebra of all bounded operators from X into X. The operator $V: X\rightarrow X$ defined by $Ve_K=e_{K+1}$ $(K=0, 1, \cdots)$ has the property that $|V|=|V^*|=1$, $V^*V=$identity, and $VV^*\neq$identity. Therefore, in a Banach algebra the product of two singular elements may be regular.

2. Let $\mathfrak{A}$ be a complex associative algebra with an identity e. By a *spectrum*, $\sigma(x)$, of an element $x\in\mathfrak{A}$ one understands the set of all complex numbers λ for which $\lambda e-x$ is not regular.

PROPOSITION 3. *For any two elements $x, y\in\mathfrak{A}$ we have*

$$\sigma(xy) \subseteq \sigma(yx) \cup \{0\}. \tag{9}$$

If $\mathfrak{A}$ is finite dimensional, then $\sigma(xy)=\sigma(yx)$

Proof. If $\lambda\notin\sigma(yx)$ and $\lambda\neq 0$, then the existence of $(\lambda e-yx)^{-1}$ implies $(\lambda e-xy)[e+x(\lambda e-yx)^{-1}y]=\lambda e-xy+\lambda x(\lambda e-yx)^{-1}y+x(-\lambda e+\lambda e-yx).(\lambda e-yx)^{-1}y=\lambda e$ and also $[e+x(\lambda e-yx)^{-1}y](\lambda e-xy)=\lambda e$. Thus $\lambda e-xy\in G$ and

$$\lambda(\lambda e - xy)^{-1} = e + x(\lambda e - yx)^{-1}y. \tag{10}$$

Thus $\lambda\notin\sigma(yx)$ and $\lambda\neq 0$ implies $\lambda\notin\sigma(xy)$; i.e., (9) is proved. If $\dim\mathfrak{A}<\infty$, then Proposition 2 and (9) imply $\sigma(xy)=\sigma(yx)$.

REMARK 4. The relation (10) relates resolvents of xy and yx and it can be obtained by the following heuristic considerations:

$$(\lambda e - xy)^{-1} = \frac{1}{\lambda}\,\frac{1}{e-\dfrac{xy}{\lambda}} = \frac{1}{\lambda}\left(e+\frac{xy}{\lambda}+\frac{xyxy}{\lambda^2}+\cdots\right)$$

$$= \frac{e}{\lambda}+\frac{x}{\lambda^2}\left(e+\frac{yx}{\lambda}+\cdots\right)y = \frac{e}{\lambda}+\frac{x}{\lambda}(\lambda e - yx)^{-1}y.$$

REMARK 5. The algebra $\mathfrak{A}$ of all matrices of order n is n^2 dimensional so that, by Proposition 3, we have $\sigma(xy)=\sigma(yx)$; i.e., matrices xy and yx have the same set of eigenvalues.

REMARK 6. Suppose that $\mathfrak{A}$ is a Banach algebra and $x,y\in\mathfrak{A}$. Assume $0\in\sigma(xy)$' but $0\notin\sigma(yx)$. Since $0\notin\sigma(yx)$, there is an $\epsilon>0$ such that $|\lambda|<\epsilon\Rightarrow\lambda\notin\sigma(yx)$ ([1]' p. 30). Hence if $0<|\lambda|<\epsilon$, then $\lambda\notin\sigma(xy)$. Thus zero is an isolated point of $\sigma(xy)$. Since the resolvent $\lambda\to(\lambda e-yx)^{-1}$ is a holomorphic function in $|\lambda|<\epsilon$, we have

$$(\lambda e - yx)^{-1} = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n + \cdots \tag{11}$$

with $a_n\in\mathfrak{A}$. From (10) and (11) for $0<|\lambda|<\epsilon$ we get

$$(\lambda e - xy)^{-1} = \frac{e+xa_0y}{\lambda} + xa_1y + xa_2y\lambda + \cdots + xa_ny\lambda^{n-1} + \cdots\text{; i.e.,}$$

$$(\lambda e-xy)^{-1}=\frac{c}{\lambda}+b_0+b_1\lambda+\cdots+b_n\lambda^n+\cdots \qquad (0<|\lambda|<\epsilon,\ b_n,\ c\in\mathfrak{A}). \tag{12}$$

From (12) we get $c\,xy=xy\,c=0$. Since yx is regular we find $cx=yc=0$. If $c\neq 0$ the resolvent $(\lambda e-xy)^{-1}$ has the first order pole in $\lambda=0$. In general, if zero is an isolated point of $\sigma(xy)$, then either $0\notin\sigma(yx)$ or zero is an isolated point of $\sigma(yx)$. Relation (10) and the way in which it was proved imply the following interesting result.

PROPOSITION 4. *Let V and V' be two complex Banach spaces. If x: $V\to V'$ and y: $V'\to V$ are two bounded operators, then $\sigma(xy)\subseteq\sigma(yx)\cup\{0\}$.*

Proof. If we denote the identity operators on V and V' by e and e' respectively, then for $\lambda \notin \sigma(yx)$ and $\lambda \neq 0$ (as in the proof of Proposition 3) we find that $\lambda e' - xy$ is regular and that (10) is to be replaced by

$$\lambda(\lambda e' - xy)^{-1} = e' + x(\lambda e - yx)^{-1}y. \tag{13}$$

For matrices

$$x = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

we have $\sigma(xy) = \{1, 2\}$ and $\sigma(yx) = \{0, 1, 2\}$. Hence for rectangular matrices x, y of types $n \times m$ and $m \times n$ respectively, in general we have $\sigma(xy) \setminus \{0\} = \sigma(yx) \setminus \{0\}$, but $\sigma(xy) \neq \sigma(yx)$.

This research was supported in part by the National Science Foundation Grant P6-1670R.

Reference

1. C. E. Rickart, General Theory of Banach Algebras, Van Nostrand, Princeton, 1960.

ON THE BEHAVIOR OF A SPECIAL SERIES

A. F. BEARDON, University of Kent at Canterbury

The two series

$$n^{-1} \sum_{j=1}^{n} \sin(j\theta) \quad \text{and} \quad n^{-1} \sum_{j=1}^{n} \cos(j\theta)$$

are easily summed and both tend to zero as $n \to \infty$. The behavior of the corresponding series

$$t_n(\theta) = n^{-1} \sum_{j=1}^{n} \tan(j\theta)$$

as $n \to \infty$ does not appear to be known and in this note we prove the following result.

THEOREM 1. (i) *Suppose that for some integers k and m, we have $k\theta = m\pi + \pi/2$. Then $t_n(\theta)$ is undefined for sufficiently large n.* (ii) *If θ/π is rational and if* (i) *is not applicable, then $t_n(\theta) \to 0$ as $n \to \infty$.* (iii) *If θ/π is irrational, then $t_n(\theta)$ does not converge to a finite limit as $n \to \infty$.*

In proving (iii) we shall use the following result.

THEOREM 2. *Let θ/π be irrational. Then n^{-1} tan $(n\theta)$ does not converge to any limit (finite or infinite) as $n \to \infty$.*

Proof. If we denote the identity operators on V and V' by e and e' respectively, then for $\lambda \notin \sigma(yx)$ and $\lambda \neq 0$ (as in the proof of Proposition 3) we find that $\lambda e' - xy$ is regular and that (10) is to be replaced by

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THEOREM 2. *Let θ/π be irrational. Then n^{-1} tan $(n\theta)$ does not converge to any limit (finite or infinite) as $n \to \infty$.*

Proof of Theorem 1. It is clear that (i) is true. To prove (ii) we suppose that $\theta = 2\pi p/q$ where $(p, q) = 1$ and where (i) is not applicable. The set $P = \{p, 2p, \cdots, qp\}$ is a complete set of residues modulo q [1, Theorem 56] and so, for some integers $k_1, \cdots, k_q$, we have $P = \{1 + k_1 q, 2 + k_2 q, \cdots, q + k_q q\}$. It follows that $\{\theta, 2\theta, \cdots, q\theta\} = \{(1 + k_1 q)2\pi/q, \cdots, (q + k_q q)2\pi/q\}$ and so

$$\sum_{j=1}^{q} \tan(j\theta) = \sum_{j=1}^{q} \tan(2\pi j/q). \tag{1}$$

Since $\tan \pi = \tan 2\pi = 0$ and $\tan(2\pi j/q) + \tan(2\pi[q-j]/q) = 0$ for $j = 1, 2, \cdots, q$ we see that (1) implies that

$$\sum_{j=1}^{q} \tan(j\theta) = 0.$$

If m is any integer, then

$$\sum_{j=1}^{q} \tan(mq + j)\theta = \sum_{j=1}^{q} \tan(j\theta) = 0$$

and so if $n = Nq + r$ with $1 \leqq r \leqq q$, then

$$|t_n(\theta)| = \left| n^{-1} \sum_{j=1}^{r} \tan(j\theta) \right| \leqq M/n$$

where

$$M = \max_r \left| \sum_{j=1}^{r} \tan(j\theta) \right| \qquad r = 1, 2, \cdots, q-1.$$

It is now clear that $t_n(\theta) \to 0$ as $n \to \infty$.

Temporarily assuming Theorem 2 we see that if $t_n(\theta) \to t$ as $n \to \infty$, then

$$\frac{\tan(n\theta)}{n} = t_n(\theta) - \frac{(n-1)}{n} t_{n-1}(\theta)$$

tends to zero as $n \to \infty$ and this contradicts Theorem 2. Thus (iii) is proved.

Proof of Theorem 2. We shall prove that $(n\theta)^{-1} \tan(n\theta)$ cannot converge. Since $\tan x$ is periodic with period π we may assume that $0 < \theta < \pi$ and we define $\delta = 1/12\pi\theta$ and observe that $\delta > 0$.

For each positive integer k let x_k be the unique solution of $\delta x = \tan x$ in the interval $k\pi < x < k\pi + \pi/2$ and let y_k be the unique solution of $\delta x = -\tan x$ in the interval $k\pi + \pi/2 < x < (k+1)\pi$. The significance of x_k and y_k is that

$$\left| \frac{\tan x}{x} \right| \geqq \delta$$

if and only if $x_k \leqq x \leqq y_k$ for some integer k. If

$$k > 1/\pi\delta\sqrt{3} = 4\sqrt{3}\theta, \tag{2}$$

then

$$\tan x_k = \delta x_k \geqq \delta\pi k > 1/\sqrt{3}$$

and so $x_k \geqq (k\pi+\pi/2)-\pi/3$. Similarly, if k satisfies (2), then $y_k \leqq (k\pi+\pi/2)+\pi/3$ and so for such k

$$\{x_k \leqq x \leqq y_k\} \subset \{\,|x - (k\pi + \pi/2)| \leqq \pi/3\}. \tag{3}$$

We next need to estimate $\{x_k \leqq x \leqq y_k\}$ from within. If k satisfies (2) and therefore (3) we have

$$\delta x_k = \tan x_k = \frac{|\sin x_k|}{|\cos x_k|} \geqq \frac{1/2}{|\cos x_k|}. \tag{4}$$

The mean value theorem implies that for all x,

$$\left|\frac{\cos x - \cos(k\pi + \pi/2)}{x - (k\pi + \pi/2)}\right| \leqq 1$$

and so for all x,

$$|\cos x| \leqq |x - (k\pi + \pi/2)|.$$

Using this together with (4) we see that

$$2\delta x_k \geqq |x_k - (k\pi + \pi/2)|^{-1}$$

and this implies that $|x_k-(k\pi+\pi/2)| \geqq 3\theta/k$. A similar estimate holds for y_k and so if k satisfies (2) then

$$\{\,|x - (k\pi + \pi/2)| < 3\theta/k\} \subset \{x_k \leqq x \leqq y_k\}.$$

From this we deduce that

$$\left|\frac{\tan(n\theta)}{n\theta}\right| \geqq \delta > 0 \tag{5}$$

if there exists an integer k satisfying (2) and $|n\theta-(k\pi+\pi/2)| < 3\theta/k$. Equivalently, k must satisfy (2) and

$$|n - k(\pi/\theta) - (\pi/2\theta)| < 3/k. \tag{6}$$

We now appeal to Theorem 440 of [1]. This states that if α is irrational, β is arbitrary, and $N>0$, then there exist integers k and n with $k>N$ and $|k\alpha-n-\beta| < 3/k$.

Taking $\alpha=\pi/\theta$ and $\beta=-\pi/2\theta$ we see that (6) holds for infinitely many pairs k_j, n_j with $k_j\to+\infty$ and $n_j\to+\infty$ as $j\to\infty$ and so (5) holds for $n=n_j$.

We next show that for some sequence of integers n_i,

$$\frac{\tan(n_i\theta)}{n_i\theta} \to 0 \quad \text{as } i \to \infty \tag{7}$$

and then the sequence $(n\theta)^{-1}\tan(n\theta)$ has at least two limit points. The proof of Theorem 2 will then be complete.

To establish (7) we note that there exist infinitely many pairs k_i, n_i with $k_i \to +\infty$, $n_i \to \infty$ as $i \to \infty$ and $|n_i(\theta/\pi)-k_i| < 1/4$ or, equivalently, $|n_i\theta - k_i\pi| < \pi/4$. This implies that $|\tan(n_i\theta)| \leqq 1$ and so

$$\frac{\tan(n_i\theta)}{n_i\theta} \to 0 \quad \text{as } i \to \infty.$$

We note that Theorem 1(iii) does not exclude the possibility that $t_n(\theta)$ converges to an infinite limit as $n \to \infty$ and it would be interesting to decide whether or not this can happen.

As a final remark, it is clear that a similar technique can be applied to the sums

$$n^{-1}\sum_{=1}^{n}\sec(j\theta),\ n^{-1}\sum_{j=1}^{n}\cot(j\theta) \quad \text{and} \quad n^{-1}\sum_{j=1}^{n}\operatorname{cosec}(j\theta).$$

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1. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 4th ed., Clarendon Press, Oxford, 1960.

THE GROUP OF THE COMPOSITION OF TWO TOURNAMENTS

BRIAN ALSPACH, Simon Fraser University, MYRON GOLDBERG
and J. W. MOON, University of Alberta

1. Introduction. A (round-robin) *tournament* T_n consists of n *nodes* $p_1, p_2, \cdots, p_n$ such that each pair of distinct nodes p_i and p_j is joined by one of the oriented *arcs* $\overrightarrow{p_ip_j}$ or $\overrightarrow{p_jp_i}$. (The nonisomorphic tournaments with three and four nodes are illustrated in Figure 1.) The notation $p_i \to p_j$ (read, p_i *dominates* p_j) means that the arc $\overrightarrow{p_ip_j}$ is in T_n. More generally, if X and Y are two disjoint subtournaments of T_n, then the notation $X \to Y$ means that every node of X dominates every node of Y.

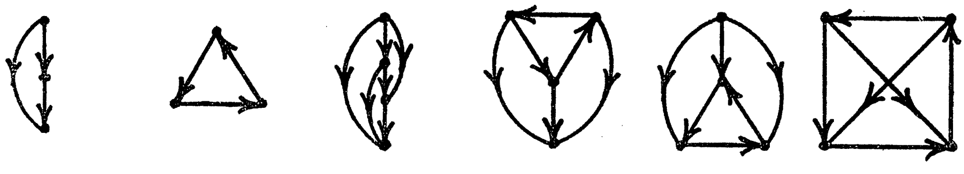

FIG. 1

Let α denote a dominance-preserving permutation of the nodes of a tournament T_n so that $\alpha(p) \to \alpha(q)$ if and only if $p \to q$. The set of all such permutations forms a group, the *automorphism group* $G(T_n)$ of T_n. It is proved in [2] that and then the sequence $(n\theta)^{-1}\tan(n\theta)$ has at least two limit points. The proof of Theorem 2 will then be complete.

To establish (7) we note that there exist infinitely many pairs k_i, n_i with $k_i \to +\infty$, $n_i \to \infty$ as $i \to \infty$ and $|n_i(\theta/\pi) - k_i| < 1/4$ or, equivalently, $|n_i\theta - k_i\pi| < \pi/4$. This implies that $|\tan(n_i\theta)| \leqq 1$ and so

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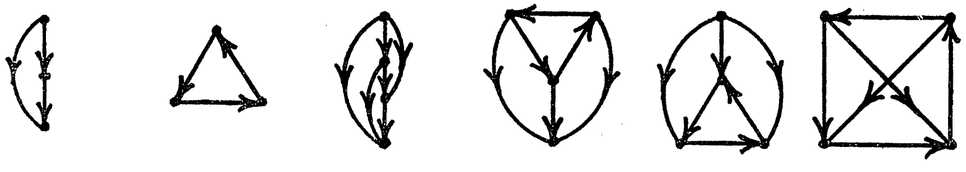

FIG. 1

Let α denote a dominance-preserving permutation of the nodes of a tournament T_n so that $\alpha(p) \to \alpha(q)$ if and only if $p \to q$. The set of all such permutations forms a group, the *automorphism group* $G(T_n)$ of T_n. It is proved in [2] that there exist tournaments whose group is abstractly isomorphic to a given group H if and only if the order of H is odd.

In Sections 2 and 3 we define the composition of two tournaments and of two groups. In Section 4 we prove that the group of the composition of two tournaments is equal to the composition of the groups of the two tournaments.

2. The composition of two tournaments. Let R and T denote two tournaments with nodes $r_1, r_2, \cdots, r_a$ and $t_1, t_2, \cdots, t_b$ respectively. The *composition* of R and T is the tournament $R \circ T$ obtained by replacing each node r_i of R by a copy $T(i)$ of T so that if $r_i \to r_j$ in R then $T(i) \to T(j)$ in $R \circ T$. More precisely, the tournament $R \circ T$ has ab nodes $p(i, k)$, where $1 \leqq i \leqq a$ and $1 \leqq k \leqq b$, such that $p(i, k) \to p(j, l)$ if and only if $r_i \to r_j$ in R or $i = j$ and $t_k \to t_l$ in T. (The composition of two 3-cycles is illustrated in Figure 2.) The composition operation was used in [1] to construct tournaments with a high degree of symmetry.

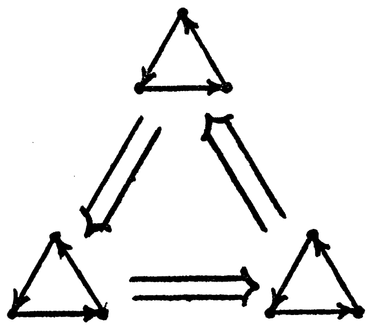

FIG. 2

3. The composition of two groups. Let F and H denote two groups with object sets U and V respectively. The *composition* (or wreath product, see [3]) of F with H is the group $F \circ H$ of all permutations α of $U \times V = \{(x, y) : x \in U, y \in V\}$ of the type

$$\alpha(x, y) = (f(x), h_x(y)),$$

where f is any element of F and h_x, for each x, is any element of H. If the objects of $U \times V$ are arranged in a matrix so that the rows and columns correspond to the objects of U and V respectively, then $F \circ H$ is the group of all permutations obtained by permuting objects in each row according to some element of H (not necessarily the same element for every row) and then permuting the rows themselves according to some element of F. If F and H have order m and n and degree u and v respectively, then $F \circ H$ has order mn^u and degree uv.

4. Theorem. We now prove that the group of two tournaments R and T is equal to the composition of the groups of R and T.

THEOREM. $G(R \circ T) = G(R) \circ G(T)$.

Proof. Consider the tournament $R \circ T$. The nodes $p(i, k)$ in each copy $T(i)$ may be permuted according to any element of $G(T)$ and the copies themselves may be permuted according to any element of $G(R)$; thus $G(R) \circ G(T)$ is certainly a subgroup of $G(R \circ T)$. To prove that the groups are the same, it will suffice to show that if a permutation α of $G(R \circ T)$ takes any node of $T(i)$ into a node of $T(j)$, then α takes every node of $T(i)$ into $T(j)$.

Suppose, on the contrary, that there exists a permutation α of $G(R \circ T)$ such that some but not all of the nodes of $\alpha(T(i))$ belong to $T(j)$. If X denotes the set of all nodes $p(i, k)$ of $T(i)$ such that $\alpha(p(i, k))$ is in $T(j)$, then X and $T(i)-X$ are both nonempty. There is no loss of generality if we assume that $i \neq j$ (if $i=j$, then α takes some subset of nodes of $T(i)-X$ into some copy $T(h)$ where $h \neq i$) and that $X \to \alpha(X)$. (If $\alpha(X) \to X$, the only alternative by the definition of $R \circ T$, then we could interchange the roles of i and j and consider the permutation α^{-1} instead.)

If any node p in $T(i)-X$ both dominates and is dominated by nodes of X, then $\alpha(p)$ has the same property with respect to the nodes of $\alpha(X)$. This implies that $\alpha(p)$ is in $T(j)$, by the definition of $R \circ T$; but then p must be in X, contrary to the hypothesis. Hence, for each node p in $T(i)-X$, either $p \to X$ or $X \to p$. One of the following two alternatives must therefore hold.

(1a) There exists a node p in $T(i)-X$ such that $p \to X$.

(1b) Every node of X dominates every node of $T(i)-X$.

Similarly, for each node q in $T(j)-\alpha(X)$ either $q \to \alpha(X)$ or $\alpha(X) \to q$. Hence each of the preceding alternatives for $T(i)$ must be considered in conjunction with one of the following alternatives for $T(j)$.

(2a) There exists a node q in $T(j)-\alpha(X)$ such that $q \to \alpha(X)$.

(2b) Every node of $\alpha(X)$ dominates every node of $T(j)-\alpha(X)$.

We first suppose that alternatives (1a) and (2a) hold.

Since $p \to X$ it follows that $\alpha(p) \to \alpha(X)$. As $\alpha(p)$ is not in $T(j)$ it must be that $\alpha(p) \to T(j)$, appealing to the definition of $R \circ T$; in particular $\alpha(p) \to q$. Similarly, $\alpha^{-1}(q) \to X$ since $q \to \alpha(X)$, and $\alpha^{-1}(q) \to T(i)$ since $\alpha^{-1}(q)$ is not in $T(i)$; in particular, $\alpha^{-1}(q) \to p$. Thus there exist two nodes, $\alpha^{-1}(q)$ and p, such that $\alpha^{-1}(q) \to p$ but $\alpha(p) \to \alpha(\alpha^{-1}(q)) = q$. This contradicts the fact that α is dominance-preserving.

We next suppose that alternatives (1a) and (2b) hold.

The *score* of a node p is the number $s(p)$ of nodes dominated by p. The score of any node $p(i, k)$ in $R \circ T$ is given by the formula

$$s(p(i, k)) = s(t_k) + b \cdot s(r_i),$$

where $s(t_k)$ and $s(r_i)$ denote the scores of t_k and r_i in T and R respectively. If $\alpha(p(i, k)) = p(j, l)$, then

$$s(p(i, k)) = s(t_k) + b \cdot s(r_i) = s(t_l) + b \cdot s(r_j) = s(p(j, l)).$$

This implies that $s(t_k) = s(t_l)$, since $0 \leqq s(t_k),\ s(t_l) < b$. Consequently

$$\sum s(t_k) = \sum s(t_l)$$

where the sums are over all k and l such that the nodes $p(i, k)$ and $p(j, l)$ are in X and $\alpha(X)$.

Let m denote the number of nodes in X and $\alpha(X)$. Then

$$\sum s(t_k) \leqq \binom{m}{2} + m(b - 1 - m)$$

by (1a) and

$$\sum s(t_l) = \binom{m}{2} + m(b - m),$$

by (2b). Therefore, the two sums cannot be equal and we have again reached a contradiction.

It remains to treat the cases when alternatives (1b) and either (2a) or (2b) hold. We shall omit the arguments for these two cases as they are quite similar to the arguments already used.

It follows from these arguments that every admissible permutation of $G(R \circ T)$ preserves the identity of the various copies $T(i)$. Since every permutation with this property belongs to $G(R) \circ G(T)$, the theorem is proved.

Sabidussi [4, 5] has proved analogous results for ordinary graphs; there the problem is complicated by the fact that there are exceptional graphs for which the corresponding result does not hold.

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IDEMPOTENT MATRICES WITH NILPOTENT DIFFERENCE

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A. Wilansky has posed the following problem [1]. Let A and B be n-square real idempotent matrices such that $(A-B)^2=0$. Then either A and B have the same range or they have the same null space. Prove that this is true for $n=2$, 3 and false for $n\geqq 4$. The submitted proof [1], due to G. N. Wollan, is incorrect for reasons given below. A correct proof appears as a special case of a more general proposition given in this note. Further, for idempotent but otherwise arbitrary A, all matrices B which are idempotent and satisfy $(A-B)^2=0$ are exhibited.

where the sums are over all k and l such that the nodes $p(i, k)$ and $p(j, l)$ are in X and $\alpha(X)$.

Let m denote the number of nodes in X and $\alpha(X)$. Then

$$\sum s(t_k) \leqq \binom{m}{2} + m(b - 1 - m)$$

by (1a) and

$$\sum s(t_l) = \binom{m}{2} + m(b - m),$$

by (2b). Therefore, the two sums cannot be equal and we have again reached a contradiction.

It remains to treat the cases when alternatives (1b) and either (2a) or (2b) hold. We shall omit the arguments for these two cases as they are quite similar to the arguments already used.

It follows from these arguments that every admissible permutation of $G(R \circ T)$ preserves the identity of the various copies $T(i)$. Since every permutation with this property belongs to $G(R) \circ G(T)$, the theorem is proved.

Sabidussi [4, 5] has proved analogous results for ordinary graphs; there the problem is complicated by the fact that there are exceptional graphs for which the corresponding result does not hold.

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For any matrix A we denote by $\rho(A)$, $R(A)$, $N(A)$, and $tr(A)$ the rank, range, null space, and trace respectively. For any two subspaces S_1 and S_2 we denote by $S_1 \cdot S_2$ their intersection and say that S_1 and S_2 are disjoint if $S_1 \cdot S_2 = 0$.

We use the terms "projection" and "idempotent matrix" interchangeably.

THEOREM 1. *Let A and B be projections of order n and $\rho(A)=r$. If $(A-B)^2=0$,* then

(i) $\rho(B)=\rho(AB)=\rho(BA)=r$

(ii) *AB and BA are projections such that* $(AB-BA)^2=0$

(iii) *$R(A)\neq R(B)$ implies $N(A)\cdot N(B)\neq 0$ and $N(A)\neq N(B)$ implies $R(A)\cdot R(B)\neq 0$.*

Proof. A nilpotent matrix has only zero roots and hence trace [2]. A projection is diagonable and has nonzero roots $\lambda=1$ [2]. Thus the trace = the rank = the number of nonzero roots (cf. [3] p. 147 and 160). Let $\rho(B)=s$. Then $r=s$. For $A-B$ is nilpotent and $tr(A-B)=tr(A)-tr(B)=0$, but $tr(A)=r$ and $tr(B)=s$. From $(A-B)^2=0$ we have

$$A+B = AB+BA \tag{1}$$

and multiplication of (1) by A and B respectively gives

$$ABA = A \quad \text{and} \tag{2}$$

$$BAB = B. \tag{3}$$

From (2), $\rho(A)=\rho(ABA)\leqq\rho(AB)\leqq\rho(A)$ and $\rho(A)=\rho(ABA)\leqq\rho(BA)\leqq\rho(A)$, which completes the proof of (i). Left multiplication of (2) by B shows $(BA)^2=BA$ and right multiplication of (2) by B shows $(AB)^2=AB$ so that AB and BA are projections. Given that AB and BA are idempotent it follows from (2) and (3) that $(AB-BA)^2=AB-B-A+BA$, which, in view of (1), is the zero matrix. Thus (ii) is proven. To prove the first implication of (iii) let $x\in R(A)$. Then $Ax=x$ and from (2) we have $ABx=Ax$. The vector $y=(I-B)x$ satisfies $Ay=By=0$. If $R(A)\neq R(B)$ there exists at least one $x\in R(A)$ such that $y\neq 0$, $y\in N(A)$, and $y\in N(B)$. For the second implication of (iii) let $u\in N(A)$; then from (1) we have $ABu=Bu$. The vector $v=Bu$ satisfies $Av=Bv=v$. If $N(A)\neq N(B)$ there exists at least one u such that $v\neq 0$, $v\in R(A)$ and $v\in R(B)$. This completes the proof of the theorem.

REMARK 1. Equations (2) and (3) define B as a reflexive generalized inverse of A and are thus known [4] to imply $\rho(A)=\rho(B)=\rho(AB)=\rho(BA)$; in fact (2) and $\rho(A)=\rho(B)$ imply (3).

REMARK 2. The content of (iii) can be restated as follows: If A and B are idempotent and such that $(A-B)^2=0$, then the ranges of A and B and the null spaces of A and B cannot both be disjoint. For, from (iii), $R(A)\cdot R(B)=0\Rightarrow N(A)=N(B)$ and $N(A)\cdot N(B)=0\Rightarrow R(A)=R(B)$. Thus under the stated conditions there always exists at least one nonzero vector which either is in both $R(A)$ and $R(B)$ or in both $N(A)$ and $N(B)$.

COROLLARY. *If A and B are idempotent matrices of order n such that $\rho(A)=r$*

and $(A-B)^2=0$, *then for* $r=1$, $n-1$ *either* $R(A)=R(B)$ *or* $N(A)=N(B)$. *In particular for* $n=2$, 3 *either* $R(A)=R(B)$ *or* $N(A)=N(B)$.

Proof. Let $r=n-1$. Then dimension $N(A)=$dimension $N(B)=1$. According to (iii), if $R(A)\neq R(B)$ there exists a nonzero vector which is in both $N(A)$ and $N(B)$ and $N(A)=N(B)$ follows. Let $r=1$; then dimension $R(A)=$dimension $R(B)=1$. By the second statement of (iii) if $N(A)\neq N(B)$ there exists a nonzero vector which is in both $R(A)$ and $R(B)$ and $R(A)=R(B)$ follows. If $n=2$, then (except in the obvious cases $r=0$, 2 which require $A=B$) we have $r=1=n-1$. If $n=3$, then (except in the obvious cases $r=0$, 3) we have either $r=1$ or $r=2=n-1$.

If $n\geqq 4$ and A and B are idempotent it does *not* follow from $(A-B)^2=0$ that A and B either have the same null space or have the same range. This can be proven by exhibiting, for $n\geqq 4$, idempotent A and B such that $(A-B)^2=0$ and such that $R(A)\neq R(B)$ and $N(A)\neq N(B)$. We know from the corollary that any such example must meet the requirement $1<\rho(A)=\rho(B)<n-1$. Particular examples are readily constructed but the following theorem shows how to obtain all possible examples.

THEOREM 2. *Let* A *be a given idempotent matrix of rank* r *where* $1\leqq r\leqq n-1$. *Then every idempotent* B *which satisfies* $(A-B)^2=0$ *is of the form*

$$B = P\begin{pmatrix} I_r & D_1 \\ D_2 & 0 \end{pmatrix}P^{-1} = PDP^{-1} \tag{4}$$

where $D_1D_2=0$, $D_2D_1=0$, *and* P *is the matrix which diagonalizes* A.

Proof. Let P be such that $P^{-1}AP=\text{diag}(I_r, 0)=C$ and define D by

$$P^{-1}BP = D = \begin{pmatrix} D_{11} & D_1 \\ D_2 & D_{22} \end{pmatrix}.$$

From (1), which is equivalent to $(A-B)^2=0$, we have $C+D=CD+DC$ from which it follows that $D_{11}=I_r$ and $D_{22}=0$. Further, $B^2=B$ implies $D^2=D$ which implies $D_1D_2=0$ and $D_2D_1=0$.

It is clear that $R(C)=R(D)$ iff $D_2=0$ and that $N(C)=N(D)$ iff $D_1=0$. With the additional observations that $N(A)\cdot N(B)\neq 0$ iff $N(C)\cdot N(D)\neq 0$ and $R(A)\cdot R(B)\neq 0$ iff $R(C)\cdot R(D)\neq 0$ the entire content of Theorem 1 and the corollary can be deduced from consideration of B as given in (4). For $n\geqq 4$ and $1<r<n-1$, every D as in (4) with $D_1\neq 0$, $D_2\neq 0$ provides a B such that $R(A)\neq R(B)$, $N(A)\neq N(B)$ and $(A-B)^2=0$.

The solution referred to in the opening paragraph [**1**] gives A with $\rho(A)=n-2$, $n\geqq 4$, and the constructed B has rank $n-1$ in conflict with (i) of Theorem 1. In fact the B of that solution is not idempotent.

The following theorem gives necessary and sufficient conditions for $(A-B)^2=0$ when A and B are projections.

THEOREM 3. *Let* A *and* B *be projections. Then each of the following sets of conditions is necessary and sufficient for* $(A-B)^2=0$:

(a) $ABA=A$ *and* $Bx\in R(A)$ *whenever* $x\in N(A)$,
(b) $BAB=B$ *and* $Ax\in R(B)$ *whenever* $x\in N(B)$.

Proof. In the proof of Theorem 1 we have seen that $(A-B)^2=0$ implies (1), (2), and (3). Now (2) is the first condition of (a) and (1) clearly implies the second condition of (a) while (3) is the first condition of (b) and (1) implies the second condition of (b). Thus (a) and (b) are both necessary. We note that every $x=(I-A)q$, where q is any vector, is in $N(A)$. If we assume (a), then $AB(I-A)q=B(I-A)q$, for arbitrary q, so that $AB-A=B-BA$. This last equality gives $(A-B)^2=0$ and thus (a) is sufficient. In the same way, every $x=(I-B)q$ is in $N(B)$ and hence if (*b*) holds, $BA(I-B)q=A(I-B)q$, for arbitrary q, which gives $BA-B=A-AB$, and thus (b) is sufficient. As the referee has noted, $(A-B)^2=(B-A)^2$; the theorem is symmetric in A and B and (b) in fact follows from (a).

The effect of requiring one or more of the projections A, B, AB and BA to be perpendicular projections is given in the following theorem. A projection E is a perpendicular projection iff it is hermitian, and iff $R(E)$ and $N(E)$ are orthogonal subspaces [**5**].

THEOREM 4. *Let A and B be idempotent and $(A-B)^2=0$. We assert:*
(a) *If A and B are hermitian, then $A=B$*
(b) *If A is hermitian, then AB is hermitian iff $N(A)=N(B)$*
(c) *If A is hermitian, then BA is hermitian iff $R(A)=R(B)$*
(d) *If A and B commute, then $A=B$.*

Proof. (a) We may write $A=B+J$ where $J^2=0$. Then A and B hermitian implies J hermitian and hence $J=0$. (b) From (i) of Theorem 1, $N(AB)=N(B)$ and $N((AB)^*)=N(A^*)$. Thus A and AB hermitian imply $N(A)=N(B)$. Conversely if $A=A^*$ and $N(A)=N(B)$ we have $N(AB)=N((AB)^*)$ which means that $R(AB)$ and $N(AB)$ are orthogonal subspaces, and AB, being a projection, is hermitian. (c) If A and BA are hermitian, then (3) reads $AB^*B=B$ which implies $R(B)=R(A)$ in view of (i) of Theorem 1. Conversely if $R(A)=R(B)$ then by (3), $R(A)=R(B)=R(BA)$. But $N(BA)=N(A)$ and $R(A)$ and $N(A)$ are orthogonal if A is hermitian. Thus $N(BA)$ and $R(BA)$ are orthogonal and BA being a projection is hermitian. (d) If $AB=BA$ then (2) and (3) read $A=AB$ and $B=AB$.

REMARK 3. Theorem 4 can be deduced from Theorem 2. Note however that the proofs given of (b) and (c) are such that these assertions hold for any A and B which satisfy (2) and (3). In fact (b) and (c) hold for any A and B satisfying (2) and (3) when the condition A hermitian is replaced by the condition A normal. For use has been made only of properties of the eigenvectors, not the roots, and the properties invoked ($N(A)=N(A^*)$, $R(A)=R(A^*)$ and $N(A)$ orthogonal to $R(A)$) are enjoyed by every normal A.

Given projections A and B such that $(A-B)^2=0$, (ii) of Theorem 1 shows the existence of two other projections with the same property. The next theorem gives a collection of results of this kind.

THEOREM 5. *Let P_1 and P_2 be projections such that $(P_1-P_2)^2=0$. Define $P_3=P_1P_2$ and $P_4=P_2P_1$ and the matrices $M_{ij}=P_i-P_j$. Then $M_{ij}^2=0$ and $M_{ij}M_{rs}=M_{rs}M_{ij}=0$ for all i, j, r, and s. Further, if P_i and P_j are hermitian, then $P_i=P_j$.*

Proof. From Theorem 1 we know that P_3 and P_4 are projections. Given $M_{12}^2=0$ we have, as in the proof of Theorem 1, $P_1+P_2=P_3+P_4$, $P_3P_1=P_1P_4=P_1$, and $P_4P_2=P_2P_3=P_2$. The first statement of the theorem is then readily verified by straightforward multiplication. If P_i and P_j are hermitian, then M_{ij} is hermitian and, being nilpotent, is the zero matrix.

We observe that $N(P_1)=N(P_2)$ is necessary and sufficient for $P_1=P_3$ and for $P_2=P_4$ and that $R(P_1)=R(P_2)$ is necessary and sufficient for $P_1=P_4$ and for $P_2=P_3$ whether or not any of the P_i are hermitian. With this observation, Theorem 5 yields alternative proofs of $(a)-(c)$ of Theorem 4.

Consider the four conditions: A is idempotent, B is idempotent, $(A-B)^2=0$, and B is a reflexive generalized inverse of A. It has been seen, in the proof of Theorem 1, that, taken together, the first three imply the fourth. The final theorem shows that, taken together, the last three imply the first.

THEOREM 6. *Let A be a projection and B any matrix such that $(A-B)^2=0$. Then B is a reflexive generalized inverse of A iff B is a projection.*

Proof. We are to show that, given $A^2=A$ and $(A-B)^2=0$, (2) and (3) hold iff $B^2=B$. The "if" part is contained in the proof of Theorem 1. Conversely if (2) holds, then from $(A-B)^2A=0$ we have $BA=B^2A$. If (3) also holds then $B=BAB=B^2AB=B^2$ follows.

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NOTE ON A PROBLEM OF ALAN SUTCLIFFE

T. J. KACZYNSKI, The University of Michigan

If n is an integer greater than 1 and $a_h, \cdots, a_1, a_0$ are nonnegative integers, let

$$(a_h, \cdots, a_1, a_0)_n \quad \text{denote} \quad a_hn^h + \cdots + a_1n + a_0.$$

Thus if $0\leqq a_i\leqq n-1$ $(i=0, \cdots, h)$, then $a_h, \cdots, a_1, a_0$ are the digits of the number $(a_h, \cdots, a_1, a_0)_n$ relative to the radix n. Alan Sutcliffe studied the problem of finding numbers that are multiplied by an integer when their digits are reversed (*Integers that are multiplied when their digits are reversed*, this MAGAZINE, THEOREM 5. *Let P_1 and P_2 be projections such that $(P_1-P_2)^2=0$. Define $P_3=P_1P_2$* and *$P_4=P_2P_1$ and the matrices $M_{ij}=P_i-P_j$. Then $M_{ij}^2=0$ and $M_{ij}M_{rs}=M_{rs}M_{ij}=0$ for all i, j, r, and s. Further, if P_i and P_j are hermitian, then $P_i=P_j$.*

Proof. From Theorem 1 we know that P_3 and P_4 are projections. Given $M_{12}^2=0$ we have, as in the proof of Theorem 1, $P_1+P_2=P_3+P_4$, $P_3P_1=P_1P_4=P_1$, and $P_4P_2=P_2P_3=P_2$. The first statement of the theorem is then readily verified by straightforward multiplication. If P_i and P_j are hermitian, then M_{ij} is hermitian and, being nilpotent, is the zero matrix.

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Consider the four conditions: A is idempotent, B is idempotent, $(A-B)^2=0$, and B is a reflexive generalized inverse of A. It has been seen, in the proof of Theorem 1, that, taken together, the first three imply the fourth. The final theorem shows that, taken together, the last three imply the first.

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Proof. We are to show that, given $A^2=A$ and $(A-B)^2=0$, (2) and (3) hold iff $B^2=B$. The "if" part is contained in the proof of Theorem 1. Conversely if (2) holds, then from $(A-B)^2A=0$ we have $BA=B^2A$. If (3) also holds then $B=BAB=B^2AB=B^2$ follows.

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If n is an integer greater than 1 and $a_h, \cdots, a_1, a_0$ are nonnegative integers, let

$$(a_h, \cdots, a_1, a_0)_n \quad \text{denote} \quad a_hn^h + \cdots + a_1n + a_0.$$

Thus if $0 \leqq a_i \leqq n-1$ $(i=0, \cdots, h)$, then $a_h, \cdots, a_1, a_0$ are the digits of the number $(a_h, \cdots, a_1, a_0)_n$ relative to the radix n. Alan Sutcliffe studied the problem of finding numbers that are multiplied by an integer when their digits are reversed (*Integers that are multiplied when their digits are reversed*, this MAGAZINE, 39 (1966) 282–287). More precisely, he looked for integers $n, a_h, \cdots, a_1, a_0, k$ satisfying the following conditions:

$$n \geqq 2, \quad k > 1, \quad a_h > 0, \quad a_0 > 0, \quad 0 \leqq a_i \leqq n-1 \qquad (i = 0, \cdots, h)$$

$$k(a_h, \cdots, a_1, a_0)_n = (a_0, a_1, \cdots, a_h)_n.$$

We shall refer to such a set of integers as an $(h+1)$-*digit solution for* n.

Sutcliffe conjectured that if there exists a 3-digit solution for n, then there also exists a 2-digit solution for n. The purpose of this paper is to prove that this conjecture is correct.

LEMMA 1. *There do not exist three natural numbers* a, c, n *satisfying the following conditions:*

$$n \geqq 2, \quad a < c \leqq n-1, \; a + c \geqq n+1, \qquad n(a-1) + c \mid n(c-1) + a.$$

Proof. Assume that a, c, n are three such natural numbers, and write

$$k(n(a-1) + c) = n(c-1) + a \qquad (k \geqq 1). \tag{1}$$

This equation implies the congruence

$$k(a - n + c) \equiv c - n + a = a - n + c \qquad (\text{mod } n-1).$$

Therefore $(k-1)\ (a-n+c) \equiv 0 \pmod{n-1}$; i.e.,

$$n - 1 \mid (k-1)(a-n+c). \tag{2}$$

Since $a+c \geqq n+1$, we see that $a-n+c>0$. If $k=1$, then $n(a-1)+c=n(c-1)+a$, so that $c-a=n(c-a)$, and hence $n=1$, contrary to our conditions. Consequently $k>1$, $(k-1)(a-n+c)>0$, and (2) implies that $(k-1)(a+c-n) \geqq n-1$. But $c-n \leqq -1$, so $(k-1)(a-1) \geqq (k-1)(a+c-n) \geqq n-1$. Therefore

$$\begin{aligned} k(n(a-1)+c) &= nk(a-1) + kc \geqq n(k-1)(a-1) + kc \geqq n(n-1) + kc \\ &\geqq nc + kc > n(c-1) + a. \end{aligned}$$

This inequality contradicts equation (1).

LEMMA 2. *If* $n=2$ or 3, *there is no* 3-*digit solution for* n.

Proof. As Sutcliffe remarked, k is always less than n, so in a solution for $n=2$, we would have $1<k<2$, which is impossible. Suppose that a, b, c, k is a 3-digit solution for $n=3$. Then $k=2$ and $2(9a+3b+c)=9c+3b+a$. It follows that $0=17a+3b-7c \geqq 17+3b-14>0$, a contradiction.

THEOREM. *Suppose* $n \geqq 2$. *If there exists a* 3-*digit solution for* n, *then there also exists a* 2-*digit solution for* n.

Proof. By Lemma 2, we may assume that $n>3$, and by Sutcliffe's Theorem 2 we may assume that $n+1$ is prime. Let $p=n+1$. Suppose that a, b, c, k is a 3-digit solution for n. Then

$$k(an^2 + bn + c) = cn^2 + bn + a. \tag{3}$$

Reducing this equation modulo p, we obtain

$$k(a - b + c) \equiv c - b + a = a - b + c \pmod{p}.$$

Thus $(k-1)(a-b+c)\equiv 0 \pmod{p}$ and $p\,|\,(k-1)(a-b+c)$. If $p\,|\,(k-1)$, then $k-1\geqq p$, which is impossible, because $k<n$. Therefore $p\,|\,(a-b+c)$. But $-p<-n<a-b+c<2n<2p$, so there are only two possibilities: either $a-b+c=0$ or $a-b+c=p$. Write $a-b+c=ep$, where e is either 0 or 1. Substituting $b=a+c-ep$ in equation (3) gives

$$k[(a(n+1) - ep)n + c(n+1)] = (c(n+1) - ep)n + a(n+1),$$

or $k[(a-e)pn+cp]=(c-e)pn+ap$.

Thus $k[(a-e)n+c]=(c-e)n+a$. If $e=1$, then $k[(a-1)n+c]=(c-1)n+a$, and $0\leqq b=a+c-p$, so that $a+c\geqq n+1$. From equation (3), $a<c$. But, by Lemma 1, this situation is impossible, so $e=0$, and $k(an+c)=cn+a$. Thus there exists a 2-digit solution for n.

COROLLARY. *If $n>3$, then there exists a 3-digit solution for n if and only if $n+1$ is not prime.*

Proof. If $n+1$ is not prime, then by Sutcliffe's Theorem 2 there exists a 2-digit solution for n, and, by Sutcliffe's Theorem 3, there must also exist a 3-digit solution. Conversely, if there exists a 3-digit solution, then there also exists a 2-digit solution and (again by Sutcliffe's Theorem 2) $n+1$ is not prime.

Note. A similar result was obtained independently by Prasert Na Nagara of Kasetsart University in Thailand.

VECTOR SPACE TECHNIQUES IN QUADRIC INVERSIONS

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In a paper of N. A. Childress [2], inversions with respect to central conics are studied. In this expository article, we shall study the problem with the use of vectors and show how the idea can be generalized.

1. Definitions and notations. We shall denote vectors by Greek letters. All vectors belong to a unitary space. The inner product of two vectors will be denoted by (ξ, ζ). We shall use standard notations of linear spaces.

According to [2], a central conic is given by the equation

$$x^2/a^2 + ky^2/b^2 = 1,$$

where $k=\pm 1$. Equations of the inversion with respect to a central conic are given as

$$(1) \qquad x' = a^2b^2x/(b^2x^2 + ka^2y^2), \qquad y' = a^2b^2y/(b^2x^2 + ka^2y^2).$$

2. A vector approach. We shall translate (1) into the language of vectors and

Reducing this equation modulo p, we obtain

$$k(a - b + c) \equiv c - b + a = a - b + c \pmod{p}.$$

Thus $(k-1)(a-b+c)\equiv 0 \pmod{p}$ and $p\mid(k-1)(a-b+c)$. If $p\mid(k-1)$, then $k-1\geqq p$, which is impossible, because $k<n$. Therefore $p\mid(a-b+c)$. But $-p<-n<a-b+c<2n<2p$, so there are only two possibilities: either $a-b+c=0$ or $a-b+c=p$. Write $a-b+c=ep$, where e is either 0 or 1. Substituting $b=a+c-ep$ in equation (3) gives

$$k[(a(n+1)-ep)n+c(n+1)] = (c(n+1)-ep)n+a(n+1),$$

or $k[(a-e)pn+cp]=(c-e)pn+ap$.

Thus $k[(a-e)n+c]=(c-e)n+a$. If $e=1$, then $k[(a-1)n+c]=(c-1)n+a$, and $0\leqq b=a+c-p$, so that $a+c\geqq n+1$. From equation (3), $a<c$. But, by Lemma 1, this situation is impossible, so $e=0$, and $k(an+c)=cn+a$. Thus there exists a 2-digit solution for n.

COROLLARY. *If $n>3$, then there exists a 3-digit solution for n if and only if $n+1$ is not prime.*

Proof. If $n+1$ is not prime, then by Sutcliffe's Theorem 2 there exists a 2-digit solution for n, and, by Sutcliffe's Theorem 3, there must also exist a 3-digit solution. Conversely, if there exists a 3-digit solution, then there also exists a 2-digit solution and (again by Sutcliffe's Theorem 2) $n+1$ is not prime.

Note. A similar result was obtained independently by Prasert Na Nagara of Kasetsart University in Thailand.

VECTOR SPACE TECHNIQUES IN QUADRIC INVERSIONS

ALI R. AMIR-MOÉZ, Texas Technological College

In a paper of N. A. Childress [2], inversions with respect to central conics are studied. In this expository article, we shall study the problem with the use of vectors and show how the idea can be generalized.

1. Definitions and notations. We shall denote vectors by Greek letters. All vectors belong to a unitary space. The inner product of two vectors will be denoted by (ξ, ζ). We shall use standard notations of linear spaces.

According to [2], a central conic is given by the equation

$$x^2/a^2 + ky^2/b^2 = 1,$$

where $k=\pm 1$. Equations of the inversion with respect to a central conic are given as

(1) $$x' = a^2b^2x/(b^2x^2 + ka^2y^2), \qquad y' = a^2b^2y/(b^2x^2 + ka^2y^2).$$

2. A vector approach. We shall translate (1) into the language of vectors and matrices. We observe that (1) can be written as

$$x' = \frac{x}{\dfrac{x^2}{a^2} + \dfrac{ky^2}{b^2}}, \qquad y' = \frac{y}{\dfrac{x^2}{a^2} + \dfrac{ky^2}{b^2}}. \tag{2}$$

We also observe that the quadratic form in the denominator can be written as

$$(x, y)\begin{bmatrix} \dfrac{1}{a^2} & 0 \\ 0 & \dfrac{k}{b^2} \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Let $\xi = (x, y)$ and let A be the matrix of this quadratic form. Then the quadratic form can be written as $(A\xi, \xi)$, and thus (2) will be

$$(x', y') = \left(\frac{x}{(A\xi, \xi)}, \frac{y}{(A\xi, \xi)}\right), \qquad (A\xi, \xi) \neq 0. \tag{3}$$

Finally we can write (3) in vector form as

$$\zeta = \frac{\xi}{(A\xi, \xi)}, \qquad (A\xi, \xi) \neq 0.$$

Indeed this vector equality is no longer restricted to the Euclidean plane. In fact we may even change the power of inversion from 1 to p, a real or complex number, or for A, use any linear transformation rather than a Hermitian one. Thus we define

$$\zeta = f(\xi) = \frac{p\xi}{(A\xi, \xi)}, \qquad (A\xi, \xi) \neq 0, \tag{4}$$

to be a quasi-inversion on a unitary space [1]. Indeed, (4) is independent of the dimension of the space. One can generalize (3) in other directions but we shall not discuss these generalizations here.

3. Properties of quasi-inversion. In [2], there is a set of theorems concerning a few invariants and transforms of a few geometric configurations. These theorems can all be generalized to a unitary space. We shall only give a sample. For example, Theorem 2 of [2] will be as follows.

THEOREM. *The transformation* (4) *transforms a hyperplane* (*coset*) *to a quadric of the form*

$$a(A\xi, \xi) + b(\xi, \delta) = 0$$

which passes through the center of inversion, i.e., the origin.

Proof. Let δ be a fixed vector in a unitary space E. Then $(\zeta, \delta) = q$, $q \neq 0$ is a hyperplane in E. We shall apply (4) to this equation and get

$$\frac{p}{(A\xi,\ \xi)}(\xi,\ \delta) - q = 0.$$

This implies that $q(A\xi,\ \xi) - p(\xi,\ \delta) = 0$. We shall leave other theorems to the reader.

References

1. A. R. Amir-Moéz, Quasi-inversion on a unitary space, Mat. Vesnik, 3, 18 (1966) 204–207.
2. N. A. Childress, Inversion with respect to the central conics, this MAGAZINE, 38 (1965) 147–149.

COMPLETELY INDEPENDENT AXIOMS FOR A SEMINATURAL SYSTEM

RICHARD A. JACOBSON, Houghton College

S. T. Stern discussed the independence, as well as the lack of complete independence, of a set of four postulates for seminatural systems [1]. Thus it is natural to pursue the quest for a completely independent set of axioms. In view of Theorem 19, [1], an obvious method of attack would be to consider the conjunction of (P-2) and (P-3) of [1] as a single axiom. This is, however, not in the true spirit of the problem. Although the resulting axiom system would be completely independent, the somewhat lengthy axiom replacing (P-2) and (P-3) does not seem to arise naturally. Therefore, it might be better to restate our problem as a quest for a completely independent set of *intuitively appealing* axioms. With that in mind we suggest that the following axiom system does indeed satisfy our criteria.

Let N be a set, R a binary relation on N, and $I(x) = \{y \in N : yRx\}$.

DEFINITION. *A set N, together with a binary relation R on N, is said to be a seminatural system if and only if the following axioms are satisfied:*

(A-1) *N is simply ordered with respect to R.*

(A-2) *N contains an element a such that for any $x \in N$ we have $I(x) \cup \{x\}$ properly contained in $I(a)'$ where $'$ denotes complement.*

(A-3) *If G is a nonempty subset of N such that for every $x \in N$, $I(x) \subset G$ implies $x \in G$, then $G = N$.*

Simple ordering implies that for any $x,\ y \in N$ exactly one of the following are true: $x = y$, xRy, or yRx.

The eight necessary examples for establishing complete independence of the axioms can be taken as numbers (3), (4), (7), (10), (13), (14), (17), (18), of [1]. The adjective *intuitively appealing* is, of course, somewhat a matter of taste and the author can only hope that each reader will find (A-2) to be acceptable in this respect.

The equivalence of the two axiom systems is evident from the following three claims:

$$\frac{p}{(A\xi,\ \xi)}(\xi,\ \delta) - q = 0.$$

This implies that $q(A\xi,\ \xi) - p(\xi,\ \delta) = 0$. We shall leave other theorems to the reader.

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CLAIM 1. (A-2)$\Rightarrow$(P-2). *Proof.* Letting $x=a$, we have $I(a)\cup\{a\}\subset I(a)'$; thus $I(a)=\phi$ and there exists no x such that xRa. Indeed $I(a)'=N$.

CLAIM 2. (A-1), (A-2)$\Rightarrow$(P-3). *Proof.* Suppose for some element x there exists no element y such that xRy. Since R is simply ordered, we have yRx if $y\neq x$. Thus $I(x)\cup\{x\}=N=I(a)'$. This is impossible and our supposition is untenable.

CLAIM 3. (P-1), (P-2), (P-3)$\Rightarrow$(A-2). *Proof.* Since there exists no x such that xRa, we have $I(a)=\phi$ and thus $I(a)'=N$. Now for any x there exists a y such that xRy. Since N is simply ordered we find that $x\neq y$ and $y\not{R}x$. Therefore, $y\notin I(x)\cup\{x\}$ and thus $I(x)\cup\{x\}$ is properly contained in $I(a)'$.

Remark 1. Example (10), in which we have ($\widetilde{\text{A-1}}$), ($\widetilde{\text{A-2}}$), (A-3), can be replaced by a simpler model; namely $N=\{a, y\}$ and yRy.

Remark 2. Four of the above examples involved finite sets. One wonders whether the axioms are still completely independent if in the original definition we require N to be an infinite set.

I wish to thank the referee for his helpful comments.

Reference

1. S. T. Stern, On the complete independence of the axioms of a seminatural system, this MAGAZINE, 39 (1966) 232–236.

ON SOME Π-HEDRAL SURFACES IN QUASI-QUASI SPACE

CLAUDE HOPPER, Omnius University

There is at present a school of mathematicians which holds that the explosive growth of jargon within mathematics is a deplorable trend. It is our purpose in this note to continue the work of Redheffer [1] in showing how terminology itself can lead to results of great elegance.

I first consolidate some results of Baker [2] and McLelland [3]. We define a class of connected snarfs as follows: $S_\alpha=\Omega(\gamma_\beta)$. Then if $B=(\otimes, \rightarrow, \theta)$ is a Boolean left subideal, we have:

$$\nabla S_\alpha = \iiint_{E(\Omega)} B(\gamma_{\beta 0}, \gamma_{\beta_0})\, d\sigma\, d\phi\, d\rho - \frac{19}{51}\Omega.$$

Rearranging, transposing, and collecting terms, we have: $\Omega=\Omega_0$.

The significance of this is obvious, for if $\{S_\alpha\}$ be a class of connected snarfs, our result shows that its union is an utterly disjoint subset of a π-hedral surface in quasi-quasi space.

We next use a result of Spyrpt [4] to derive a property of wild cells in door topologies. Let ξ be the null operator on a door topology, $\square$, which is a superlinear space. Let $\{P_\gamma\}$ be the collection of all nonvoid, closed, convex, bounded, compact, circled, symmetric, connected, central, Z-directed, meager sets in $\square$.

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Then $P = \cup P_\gamma$ is perfect. Moreover, if $P \neq \phi$, then P is superb.

Proof. The proof uses a lemma due to Sriniswamiramanathan [5]. This states that any unbounded fantastic set is closed. Hence we have

$$\Rightarrow P \sim \xi(P_\gamma) - \tfrac{1}{3}.$$

After some manipulation we obtain

$$\tfrac{1}{3} = \tfrac{1}{3}$$

I have reason to believe [6] that this implies P is perfect. If $P \neq \phi$, P is superb. Moreover, if $\square$ is a T_2 space, P is simply superb. This completes the proof.

Our final result is a generalization of a theorem of Tz, [7] and encompasses some comments on the work of Beaman [8] on the Jolly function.

Let Ω be any π-hedral surface in a semi-quasi space. Define a nonnegative, nonnegatively homogeneous subadditive linear functional f on $X \supset \Omega$ such that f violently suppresses Ω. Then f is the Jolly function.

Proof. Suppose f is not the Jolly function. Then $\{\Lambda, @, \xi\} \cap \{\Delta, \Omega, \Rightarrow\}$ is void. Hence f is morbid. This is a contradiction, of course. Therefore, f is the Jolly function. Moreover, if Ω is a circled husk, and Δ is a pointed spear, then f is uproarious.

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A CALCULUS FALLACY

K. W. MILLER, IIT Research Institute

Given

$$y = ae^{ix} \tag{1}$$

where a is a constant, then

$$\frac{dy}{dx} = iae^{ix} = iy \tag{2}$$

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Combining (1), (2), and (3) we have

$$\frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2. \tag{4}$$

Integrating once we obtain

$$\frac{dy}{dx} = \int \frac{1}{y}\left(\frac{dy}{dx}\right)^2 dx + C_1 = \int\left(\frac{1}{y}\,\frac{dy}{dx}\right)dy + C_1. \tag{5}$$

Integrating again we have

$$y = \int\left[\int\left(\frac{1}{y}\,\frac{dy}{dx}\right)dy\right]dx + C_1x - C_2 \tag{6}$$

where C_1 and C_2 are integration constants.

Now changing the order of integration we get

$$\begin{aligned} y &= \int\left[\int\left(\frac{1}{y}\,\frac{dy}{dx}\right)dx\right]dy + C_1x - C_2 \\ &= \int\left[\int\frac{dy}{y}\right]dy + C_1x - C_2 \\ &= \int [\ln y]dy + C_1x - C_2 \\ &= y\ln y - y + C_1x - C_2. \end{aligned}$$

Rearranging terms we obtain

$$x = [y(2 - \ln y) + C_2]/C_1 = C_3y(2 - \ln y) + C_4. \tag{7}$$

However, solving (1) for x, we have

$$x = (1/i)\ln(y/a) = i(\ln a - \ln y). \tag{8}$$

But (7) and (8) are *not* equivalent. Where is the fallacy?!

CONCERNING THE JORDAN NORMAL FORM

RICHARD SINKHORN, University of Houston

Every $n \times n$ complex matrix A is similar to a matrix J which is the direct sum of matrices of the form and

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$$J_k = \begin{bmatrix} \lambda & 1 & 0 \cdots 0 & 0 \\ 0 & \lambda & 1 \cdots 0 & 0 \\ 0 & 0 & \lambda \cdots 0 & 0 \\ \cdot & \cdot & \cdot\cdot\cdot\cdot\cdot & \cdot \\ 0 & 0 & 0 \cdots \lambda & 1 \\ 0 & 0 & 0 \cdots 0 & \lambda \end{bmatrix}.$$

J is called the Jordan normal form of A; the J_k are called Jordan blocks. It is the intent of this paper to derive an algorithm for determining the Jordan form of a given matrix by applying methods of the finite difference calculus. We adopt the usual notion of first and higher differences: $\Delta_x F(x) = \Delta_x^1 F(x) = F(x+1) - F(x)$, $\Delta_x^n F(x) = \Delta_x[\Delta_x^{n-1} F(x)]$, $n > 1$. We also define A^0 to be $n \times n$ identity matrix for any $n \times n$ matrix A. By ker A we mean the set $\{v \mid Av = 0\}$.

THEOREM. *Let A have Jordan form J. The number of $p \times p$ Jordan blocks in J corresponding to an eigenvalue λ is* $-\Delta_x^2 \dim \ker(A - I\lambda)^x|_{x=p-1}$.

Proof. Suppose J contains m Jordan blocks corresponding to a root λ. Then $J - I\lambda$ contains one row of zeros for each block and it follows that

$$\dim \ker (J - I\lambda) = m.$$

Suppose that J contains m_p $p \times p$ Jordan blocks corresponding to λ. There are

$$m - \sum_{k=1}^{p} m_k$$

blocks corresponding to λ of order greater than p.

For any Jordan block J_k of order less than or equal to p, $(J_k - I\lambda)^p = (J_k - I\lambda)^{p+1} = 0$. For any J_k of order greater than p, there is one more row of zeros in $(J_k - I\lambda)^{p+1}$ than there is in $(J_k - I\lambda)^p$. It follows that the number of J_k of order larger than p is

$$\dim \ker (J - I\lambda)^{p+1} - \dim \ker (J - I\lambda)^p = \Delta_x \dim \ker (J - I\lambda)^x|_{x=p};$$

whence

$$m - \sum_{k=1}^{p} m_k = \Delta_x \dim \ker (J - I\lambda)^x|_{x=p}.$$

From

$$m - \sum_{k=1}^{p-1} m_k = \Delta_x \dim \ker (J - I\lambda)^x|_{x=p-1}$$

we have

$$m_p = \Delta_x \dim \ker (J - I\lambda)^x|_{x=p-1} - \Delta_x \dim \ker (J - I\lambda)^x|_{x=p}$$

$$= -\Delta_x^2 \dim \ker (J - I\lambda)^x|_{x=p-1}.$$

Since $\dim \ker(A - I\lambda)^x = \dim \ker(J - I\lambda)^x$, the theorem follows.

Example.

$$A = \begin{pmatrix} 5 & -1 & -3 & 2 & -5 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & -2 \\ 0 & -1 & 0 & 3 & 1 \\ 1 & -1 & -1 & 1 & 1 \end{pmatrix}.$$

The characteristic equation of A is $f(x) = (x-2)^3(x-3)^2$. We compose two difference tables for the roots 2 and 3, respectively.

$\lambda = 2$

x	$\dim \ker (A - 2I)^x$	Δ_x	Δ_x^2
0	0		
		2	
1 ←	2		−1
		1	
2 ←	3		−1
		0	
3	3		

$\lambda = 3$

x	$\dim \ker (A - 3I)^x$	Δ_x	Δ_x^2
0	0		
		1	
1 ←	1		0
		1	
2 ←	2		−1
		0	
3	2		

We terminate each difference table at the least value for x for which $\dim \ker(A - I\lambda)^x = \dim \ker(A - I\lambda)^{x-1}$ since then $\dim \ker(A - I\lambda)^p$ is this same value for all $p \geqq x-1$. The Δ_x^2 columns tell us that there is one 1×1 block and one 2×2 block corresponding to the root 2; there are no 1×1 blocks and there is one 2×2 block corresponding to the root 3. Thus the Jordan form of A is

$$J = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

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PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles, California 90029.

To be considered for publication, solutions should be mailed before June 1, 1968.

PROPOSALS

684. *Proposed by J. David Martin, Washington State University.*

Define a C-number as a number having a common factor with all nonprime numbers smaller than itself. List all C-numbers, and prove that the list is exhaustive.

685. *Proposed by Jack M. Elkin, Polytechnic Institute of Brooklyn.*

Prove Fermat's Principle for a circular mirror. That is, given two points, A and B, inside a circle, locate P such that $AP+PB$ is an extremum.

686. *Proposed by Stanley E. Payne, Miami University, Ohio.*

For each positive integer $m>1$ define the polynomial $f_m(x)$ by $f_m(x) = \sum\phi(d)x^{m/d}$, where ϕ is Euler's phi function and the sum is extended over all positive divisors of m. For any integer n, prove that

$$f_m(n) \equiv 0 \pmod{m}.$$

687. *Proposed by Sidney H. L. Kung, Jacksonville University, Florida.*

Prove that if the perimeter of a quadrilateral $ABCD$ is cut into two portions of equal length by all straight lines passing through a fixed point O in it, the quadrilateral is a parallelogram.

688. *Proposed by L. Carlitz, Duke University.*

Let n be an odd positive integer, a an arbitrary integer prime to n. Show that

$$\sum_{s=1}^{(n-1)/2} [1/2 + as/n] \equiv 0 \pmod{2}$$

provided a is odd or $n \equiv \pm 1 \pmod{8}$.

689. *Proposed by Alexandru Lupas, Institutul De Calcul, Cluj, Romania.*

Let the lengths of the sides of a triangle be $a_1 \geqq a_2 \geqq a_3$. Also let r and R, respectively, denote the radii of the inscribed and circumscribed circle of an arbitrary triangle with angles B_i, $(0, \pi/2)$, $i=1, 2, 3$. Then the following inequality holds

$$\sum_{i=1}^{3} (a_{i+1}a_{i+2})^{\cos B_i} \leqq 2 + 1/2 \sum_{i=1}^{3} a_i^2 - r/R$$

equality holding only when $a_1=a_2=a_3=1$ and $B_1=B_2=B_3$.

690. *Proposed by J. M. Gandhi, University of Alberta, Canada.*

Prove that

$$\sum_{j=0}^{r-1}\sum_{i=0}^{j} j!/(j-1)$$

$$= \begin{vmatrix} 0 & -\binom{1}{1} & 0 & 0 & \cdots & 0 \\ 1! & +\binom{2}{1} & -\binom{2}{2} & 0 & \cdots & 0 \\ 2! & -\binom{3}{1} & +\binom{3}{2} & -\binom{3}{3} & \cdots & 0 \\ — & — & — & — & \cdots & — \\ (\gamma-2)! & — & — & - & \cdots & \binom{\gamma-1}{\gamma-1} \\ (\gamma-1)! & -1^{\gamma}\binom{\gamma}{1} & +(-1)^{\gamma-1}\binom{\gamma}{2} & - & \cdots & \binom{\gamma}{\gamma-1} \end{vmatrix}$$

SOLUTIONS

Late Solutions

David Fettner, City College of New York: **648**; *Richard A. Jacobson, Houghton College, New York:* **655**; *Lew Kowarski, Morgan State College, Maryland:* **662**; *Louis Sass, Carnegie Institute of Technology:* **652.**

Pythagorean Triples

663. [September, 1967] *Proposed by Charles W. Trigg, San Diego, California.*

Find a Pythagorean triangle with sides of three digits each such that the nine digits involved are distinct.

Solution by Michael Goldberg, Washington, D.C.

If $a^2+b^2=c^2$, then all primitive Pythagorean triplets are obtained as $a, b, c = 2pq, p^2-q^2, p^2+q^2$, where p and q are integers. By trial, the primitive solutions do not meet the conditions of the problem. However, multiples of the primitive solutions may yield satisfactory solutions. One such solution is $182(3, 4, 5)=546, 728, 910$. Another solution is $178(3, 4, 5)=534, 712, 890$.

Also solved by V. C. Bailey, University of Evansville, Indiana (one solution); Leon Bankoff, Los Angeles, California; J. A. H. Hunter, Toronto, Ontario, Canada; Lew Kowarski, Morgan State College, Maryland (one solution); Sam Kravitz, East Cleveland, Ohio; James G. Seiler, San Diego City College; James N. Webb, W. H. Harrison High School, Evansville, Indiana; Charles B. Weinberg, Rockville Center, New York (one solution) and the proposer.

A Convergent Sequence

664. [September, 1967] *Proposed by Bruce B. Peterson, Middlebury College, Vermont.*

If $\{p_n/q_n\}$ is a sequence of distinct rationals which converges to x and $\{s_n\}$ is a bounded sequence, then the sequence $(p_n+s_n)/q_n$ converges to x.

Solution by Lawrence A. Ringenberg, Eastern Illinois University.

$$\text{For } n = 1, 2, 3, \cdots \text{ let } p_n = 1 + 1/n, \quad q_n = 1, \quad s_n = 1.$$

Then $\{(p_n/q_n)\}$ is a sequence of distinct rationals that converges to 1, $\{s_n\}$ is a bounded sequence, and $\{(p_n+s_n)/q_n\}$ converges to 2. This shows that the proposed statement is false. Following is a proof of the statement modified to require that p_n, q_n, s_n be integers and that x be a real number.

Since $\{(p_n/q_n)\}$ converges, it is a bounded sequence. It follows that for any given denominator q_n there are at most a finite number of distinct terms in the sequence $\{(p_n/q_n)\}$ with that denominator. Therefore $q_n \to \infty_{n\to\infty}$ and $(p_n+s_n)/q_n = (p_n+q_n)+(s_n/q_n) \to_{n\to\infty} x+0=x$.

Also solved by Donald Batman, M.I.T. Lincoln Laboratory; Richard A. Bell, Gustavus Adolphus College, Minnesota; Wray G. Brady, University of Bridgeport; Nicholas C. Bystrom, Northland College, Wisconsin; E. M. Clarke, Duke University; George Fabian, Park Forest, Illinois; Michael Goldberg, Washington, D. C.; Richard A. Jacobson, Houghton College, New York; J. F. Leetch, Bowling Green State University, Ohio; R. S. Luthar, University of Wisconsin at Waukesha; John J. Moore, Niagara Falls, New York; Stanley Rabinowitz, Far Rockaway, New York; Kenneth A. Ribet, Brown University; Dimitrios Vathis, Chalcis, Greece; and the proposer.

Each of the above made an assumption regarding q_n which restricted it from taking on any particular value more than a finite number of times.

Rosalind Nelson, Catlin Gabel High School, Portland, Oregon, and Erwin Just, Bronx Community College, each pointed out another counterexample when $\{q_n\}=1$.

Constant Hypervolume

665. [September, 1967] *Proposed by Gregory Wulczyn, Bucknell University, Pennsylvania.*

Show that the volume of the hypersolid formed by the n coordinate planes and the tangent plane at any point of the hypersurface $x_1x_2x_3 \cdots x_n - a^n = 0$ is a constant.

Solution by John F. Burke, St. Lawrence University, New York.

If the coordinates of the point of contact of the tangent plane to the hypersurface $x_1x_2x_3 \cdots x_n - a^n = 0$ are $(u_1, u_2, u_3, \cdots, u_n)$ then the equation of the tangent plane is given by

$(x_1-u_1)u_2u_3\cdots u_n+(x_2-u_2)u_1u_3\cdots u_n+\cdots+(x_j-u_j)$
$u_1u_2\cdots u_{j-1}u_{j+1}\cdots u_n+\cdots+(x_n-u_n)u_1u_2\cdots u_{n-1}=0.$

The hypersolid formed by the tangent plane and the n coordinate planes is in fact an n dimensional simplex with vertices at the origin and at the points of intersection of the tangent plane with the n coordinate axes. Hence the volume V of the hypersolid is given by the $n+1$ order determinant

$$V=\frac{1}{n!}\begin{vmatrix} \frac{na^n}{u_2u_3\cdots u_n} & 0 & \cdots & 0 & 1 \\ 0 & \frac{na^n}{u_1u_3\cdots u_n} & \cdots & 0 & 1 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{na^n}{u_1u_2\cdots u_{n-1}} & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix}$$

$=n^na^n/n!$ which is clearly a constant.

Also solved by V. C. Bailey, University of Evansville, Indiana; Michael Goldberg, Washington, D.C.; Richard Laatsch, Miami University, Ohio; and the proposer.

Partitions of a Rectangle

666. [September, 1967] *Proposed by Dennis P. Geller, University of Michigan.*

Consider a rectangle R with sides x and y, $x<y$. Suppose that we remove from R an x by x square to get a rectangle R'. If R' is similar to R, we know that R is the golden rectangle and $y/x=(1+\sqrt{5})/2$. We will suppose, however, that R' is not similar to R. Removing as before a square with side the smaller of the two sides of R', we have a rectangle R''. Under what conditions is R'' similar to R?

Solution by John Hudson Tiner, Harrisburg, Arkansas.

If $x<y-x$ and R is similar to R'', then

$$\frac{y}{x}=\frac{x}{y-2x}\cdot$$

Or,

$$\frac{y}{x}=\frac{1+\sqrt{2}}{1}\cdot$$

If $y-x<x$ and R is similar to R'' and the long side of R is parallel to the long side of R'', then

$$\frac{y}{x}=\frac{y-x}{2x-y}\cdot$$

Or,

$$\frac{y}{x}=\frac{1+\sqrt{5}}{2}.$$

In this case, however, the condition that R is not similar to R' is not met.

If $y-x<x$ and R is similar to R'' and the long side of R is parallel to the short side of R'', then

$$\frac{y}{x}=\frac{2x-y}{y-x}.$$

Or,

$$\frac{y}{x}=\frac{\sqrt{2}}{1}.$$

Also solved by V. C. Bailey, University of Evansville, Indiana; Leon Bankoff, Los Angeles, California; Farrell Bloch, Swarthmore College; Henry Frandsen, University of Tennessee; Herta T. Freitag, Hollins College, Virginia; J. Ray Hanna, University of Wyoming; Richard A. Jacobson, Houghton College, New York; Lew Kowarski, Morgan State College, Maryland; James R. Kuttler and Nathan Rubinstein (Jointly) (partial solution); Johns Hopkins University, Applied Physics Laboratory; Herbert R. Leifer, Pittsburgh, Pa.; Sister Stephanie Sloyan, Georgian Court College; Stanley Rabinowitz, Far Rockaway, New York; David R. Stark, V. A. Hospital, Cleveland, Ohio; Charles W. Trigg, San Diego, California; Dimitrios Vathis, Chalcis, Greece; Dale Woods and Robert Dowers, Kirksville, Missouri; Gregory Wulczyn, Bucknell University, Pennsylvania; and the proposer. One incorrect solution was received.

A Well-Known Derangement

667. [September, 1967] *Proposed by Lew Kowarski, Morgan State College, Maryland.*

In how many different ways can one place on a shelf N encyclopedia volumes so that no volume is in its proper place?

Solution by Stanley Rabinowitz, Far Rockaway, New York.

The number of ways is merely the number of derangements of n objects. This is a classic result and is known to be

$$!n = n \text{ subfactorial} = n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^n\frac{1}{n!}\right].$$

The problem has been proposed in many forms. I have found it in the following sources:

Ball and Coxeter, *Mathematical Recreations and Essays*, P. 46.
Chrystal, *Textbook of Algebra*, Part Two, P. 24.
Dorrie, *100 Great Problems of Elementary Mathematics*, Problem 6.
Dudeney, *Amusements in Mathematics*, Problem 267.
Graham, *Ingenious Mathematical Problems and Methods*, Problem 26.

Niven, *Mathematics of Choice*, P. 78.
Ryser, *Combinatorial Mathematics*, P. 22.

The problem is also equivalent to Problem 5 (Part I) of the Putnam Contest, November, 1958.

Solutions and references also submitted by Leon Bankoff, Los Angeles, California; Donald Batman, M.I.T. Lincoln Laboratory; Wray G. Brady, University of Bridgeport; Brother Alfred Brousseau, Saint Mary's College, California; William Farmer and Yvonne Yuen, Colorado State University; Anton Glaser, Pennsylvania State University, Abington, Pennsylvania; D. Gootkin, AVCO Missile System; James C. Hickman, University of Iowa; Gerald G. Jahn, Wisconsin State University; Erwin Just, Bronx Community College; James R. Kuttler and Nathan Rubinstein, Johns Hopkins University, Applied Physics Laboratory (Jointly); Richard Laatsch, Miami University, Ohio; J. F. Leetch, Bowling Green State University, Ohio; Douglas Lind, University of Virginia; John E. McDonald, Wappingers Falls, New York; William Moser, McGill University; C. Stanley Ogilvy, Hamilton College, New York; Stephen K. Prothero, Willamette University, Oregon; Daniel R. Stark, V. A. Hospital, Cleveland, Ohio; Gary E. Stevens, Bowling Green State University; Henry J. Ricardo, Manhattan College; Charles W. Trigg, San Diego, California; Dale Woods and Claude E. Smith, Kirksville, Missouri; and the proposer. Two incorrect solutions were received.

The Pancake Problem

668. [September, 1967] *Proposed by Walter W. Horner, Pittsburgh, Pennsylvania.*

A cook is making pancakes on a circular griddle 26 units in diameter. She poured batter for three circular pancakes of unequal size which completely covered a diameter of the griddle but only half of its area. Find the diameters of the pancakes in integers.

Solution by Brother Alfred Brousseau, Saint Mary's College, California.

Since the areas are proportional to the squares of respective diameters, it is sufficient to find three distinct integers whose sum is 26 and the sum of whose squares is $676/2=338$. These are 16, 9, and 1.

GENERALIZATION. Let x, y, z be the diameters of the pancakes. Then

$$x^2+y^2+z^2=(x+y+z)^2/2$$

This leads to the Diophantine equation:

$$x^2+y^2+z^2-2xy-2xz-2yz=0$$

or

$$(x-y-z)^2=4yz$$

for which the positive solution is:

$$x=y+z+2\sqrt{yz}$$

The problem can be solved for any set of values yz which is a perfect square.

Also solved by V. C. Bailey, University of Evansville, Indiana; Francine Bankoff, Los Angeles, California; Merrill Barnebey, Wisconsin State University at La Crosse; Farrell Bloch, Swarthmore College; J. W. Bloomsburg, Lewis and Clark Normal School, Lewiston, Idaho; Fred E. Ermis, Jr., Wharton County Junior College, Texas; George Fabian, Park Forest, Illinois; Anton Glaser, Pennsylvania State University, Abington, Pennsylvania; Michael Goldberg, Washington, D. C.; John E.

Homer, Union Carbide Corporation, Chicago; J. A. H. Hunter, Toronto, Ontario, Canada; L. A. Jacobson MIT Lincoln Laboratory; R. A. Jacobson, Houghton College, New York; James R. Kuttler and Nathan Rubinstein, Johns Hopkins University, Applied Physics Laboratory (Jointly); Richard Laatsch, Miami University, Ohio; J. F. Leetch, Bowling Green State University, Ohio; Herbert R. Leifer, Pittsburgh, Pennsylvania; Joseph S. Madachy, Kettering, Ohio; C. Stanley Ogilvy, Hamilton College, New York; C. C. Oursler, Southern Illinois University; Stanley Rabinowitz, Far Rockaway, New York; Melvin Rein, Morton, Illinois; Marilyn Rodeen, San Francisco, California; Louis Sass, Carnegie Institute of Technology; Sister Stephenie Sloyan, Georgian Court College; Daniel R. Stark, V. A. Hospital, Cleveland, Ohio; E. P. Starke, Plainfield, New Jersey; John Judson Tiner, Harrisburg, Arkansas; Charles W. Trigg, San Diego, California; Craig A. Vogel, Concordia Teachers College, Nebraska; P. Weygang, Levack District High School, Ontario, Canada; Henry Wimble, Michigan Technological University; Dale Woods and Ray Robinson, Kirksville, Missouri; Gregory Wulczyn, Bucknell University; and the proposer.

Comment on Q394

Q394. [November, 1966, and September, 1967] Given a line segment AB of length $2k$. Find the area of the plane ring whose outer circle goes through A and B, and inner circle is tangent to AB.

[*Submitted by Vladimir F. Ivanoff*]

Comment by Aubrey J. Kempner, University of Colorado.

The generalization in the comment on Q394 seems too good to be true. Apply it to the ellipse $x^2/a^2+y^2/b^2=1$ where $a>b$, $2k=2a$.

Comment on Q415

Q415. [September, 1967] Show that if A is an n square matrix and each row (column) sums to c, then c is a characteristic root of A.

[*Submitted by Clarence C. Morrison*]

Comment by A. Wilansky, Lehigh University.

A shorter and more elementary solution is contained in the following.

From the statement of the problem we have $A1=c1$ where

$$1=\begin{pmatrix}1\\1\\1\\ \cdot\\ \cdot\\ \cdot\\1\end{pmatrix}$$

Therefore c is a characteristic root of A.

Errata

On Page 165, May, 1967, in the solvers of Problem 637, the name Dimitrios Valthis should read Dimitrios Vathis.

On Page 199, September, 1967, in A415, the words rows and columns should be interchanged.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q425. The following question and answer appeared in a mathematical magazine: Question: "A carpenter wishes to cut a wooden cube three inches on a side into 27 one-inch cubes. He can do this easily by making six cuts through the cube keeping the pieces together in the cube shape. What is the minimum possible number of cuts if the carpenter is permitted to rearrange the pieces in any desired manner after each cut?" Answer: "Six cuts, since the interior cube requires one cut along each of its six faces."

If the interior cube was missing, what is the minimum possible number of cuts if the carpenter is permitted to rearrange pieces in any desired manner after each cut?

[*Submitted by Sam Newman*]

Q426. Without using calculus, determine the least value of the function $f(x) = (x+a+b)(x+a-b)(x-a+b)(x-a-b)$, where a and b are real constants.

[*Submitted by Roger B. Eggleton*]

Q427. Define

$$T_k(x) = \prod_{\substack{i=1 \\ i \neq k}}^{n} (x - i)$$

and let $P(x)$ be a polynomial of minimum degree in which $P(k) = mT_k(k)$, $(k = 1, 2, \cdots, n)$ for some constant m. If s and t are both integers such that $1 \leqq s \leqq n$ and $1 \leqq t \leqq n$ prove that

$$\int_s^t P(x)dx = 0.$$

[*Submitted by Erwin Just*]

Q428. Let h be a homomorphism from the ring $R[x]$ of real polynomials into the real field R such that $h(1) = 1$. Show that there exists a real number r_h such that $h(p) = p(r_h)$ for all p in $R[x]$.

[*Submitted by Lyle E. Pursell*]

Q429. Suppose $f(re^{i\phi}) = u(r, \phi) + iv(r, \phi)$ is analytic in $|z| < 1$, $f(0)$ real. Show that

$$\int_0^{2\pi} v^2(r, \phi)d\phi \leqq \int_0^{2\pi} u^2(r, \phi)d\phi$$

for $r < 1$.

[*Submitted by Walter O. Egerland*]

(Answers on page 94)

ANSWERS

A425. After the first cut, there are two parts. The larger of these parts (composed of 17 one-inch cubes) has a central one-inch cube whose four faces will require a minimum of four cuts. After the last of these cuts, regardless of any rearrangements, there will remain at least two one-inch cubes requiring one more cut. The total will still be six cuts.

A426. Clearly we may take $a \geqq b \geqq 0$ without loss of generality. Then if Δ denotes the area of a triangle with sides $2a$, $2b$, and $2x$ we have $f(x) = -\Delta^2$. Since Δ is maximum when the angle between the two sides of fixed length $2a$ and $2b$ is a right angle, this occurs when $x^2 = a^2 + b^2$ and

$$\Delta_{\max} = \tfrac{1}{2}(2a)(2b) = 2ab.$$

Thus $f_{\min} = -4a^2b^2$.

A427. It is readily verified that

$$P(x) = m \sum_{k=1}^{n} T_k(x)$$

and

$$\int_s^t P(x)dx = m \prod_{i=1}^{n} (x - i) \Big|_s^t,$$

from which the conclusion follows.

A428. It is well known that the only nonzero homomorphism from R into R is the identity. Hence h acting on the subring of constant polynomials is the identity. Now if

$$p(x) = \sum a_n x^n,$$

then

$$\begin{aligned} h(p) &= \sum h(a_n)[h(x)]^n \\ &= \sum a_n[h(x)]^n = p[h(x)]. \end{aligned}$$

Therefore $h(x)$ is the desired number r_h.

A429. $\operatorname{Re} f^2(z) = u^2 - v^2$ is harmonic in $|z| < 1$ and by Gauss' mean value theorem

$$\int_0^{2\pi} (u^2 - v^2)d\phi = 2\pi u^2(0) \geqq 0$$

from which the proposition follows.

(Quickies on page 102)

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1967 — *$7.50*

Department CC

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MATHEMATICS: The Alphabet of Science

By MARGARET F. WILLERDING, *San Diego State College; and* RUTH A. HAYWARD, *General Dynamics, Convair Division.* The general reader or student with little or no mathematics background can read this text with understanding and gain an appreciation of the beauty and scope of mathematics. The explanations are detailed and clear. The topics are simple yet profound. They have applications in other fields, and are selected from both classical and modern mathematics. Most of the chapters are completely independent and may be studied according to the interest of the reader. The book enables him to develop insight into logic, number theory, mathematical systems, probability, matrix algebra, computer programming and other exciting topics.

1968. In press.

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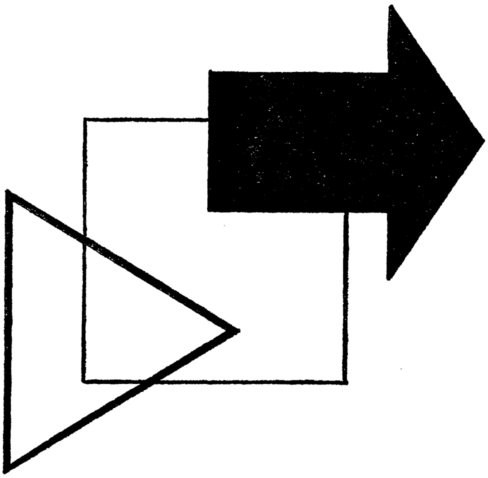

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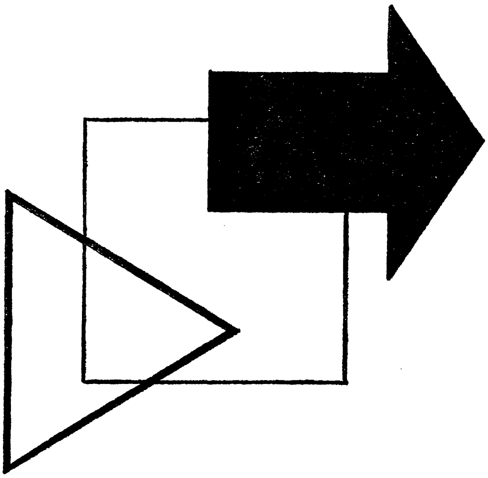

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